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Second-gradient homogenized model for wave propagation in heterogeneous periodic media

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ABSTRACT

The paper is focused on a homogenization procedure for the analysis of wave propagation in materials with periodic microstructure. By a reformulation of the variational-asymptotic homogenization technique recently proposed by Bacigalupo and Gambarotta (2012a), a second-gradient continuum model is derived, which provides a sufficiently accurate approximation of the lowest (acoustic) branch of the dispersion curves obtained by the Floquet–Bloch theory and may be a useful tool for the wave propagation analysis in bounded domains. The multi-scale kinematics is described through micro-fluctuation functions of the displacement field, which are derived by the solution of a recurrent sequence of cell BVPs and obtained as the superposition of a static and dynamic contribution. The latters are proportional to the even powers of the phase velocity and consequently the micro-fluctuation functions also depend on the direction of propagation. Therefore, both the higher order elastic moduli and the inertial terms result to depend by the dynamic correctors. This approach is applied to the study of wave propagation in layered bi-materials with orthotropic phases, having an axis of orthotropy parallel to the direction of layering, in which case, the overall elastic and inertial constants can be determined analytically. The reliability of the proposed procedure is analysed by comparing the obtained dispersion functions with those derived by the Floquet–Bloch theory.

1. Introduction

The recognition that the microstructure in heterogeneous materials affects the dispersive propagation of elastic waves dates back to the seminal paper of Mindlin (1964) in which several non-local continuum models to describe the dispersion relations of plane waves were proposed. As is well known this approach was based on a phenomenological description of the microstructure effects in terms of appropriate representations of the micro-deformation according to either micromorphic (see Berezovski et al., 2013) or second gradient continuum models (see Papargyri-Beskou et al., 2009) to obtain the frequency spectrum, i.e. the dispersion function related to bulk elastic waves. Nevertheless, these approaches are usually formulated at the macroscale and the material parameters involved in the description of wave propagation have a phenomenological character and need to be calibrated from simulation of experiments. To circumvent this problem, multi-scale methods are needed to relate the material microstructure and the dispersive properties of wave propagation.

For some decades, the study of layered materials and more generally of materials having periodic microstructure has deserved a remarkable interest, with various applications ranging from seismology up to micro and nano-devices. For layered materials, dispersion relations were obtained analytically by Rytov (1956), and Sun et al. (1968), through approaches based on the Floquet–Bloch theory. This theory, which implies the solution of a periodic cell with prescribed specific boundary conditions, provides an exact description of the Bloch mode spectrum (i.e. the frequency-wave number dispersion relations), represented by several branches defining passing and stopping frequency bands. Within this approach different solution strategies have been developed. Kohn et al. (1972) and Nemat-Nasser (1972) tackled this problem by formulating both displacement and mixed variational methods. Joao and Siems (1994) developed a variational approach based on proper non-smooth-expansion functions to take into account the discontinuity of the material constants at the interface between inclusion and matrix. Several approaches based on a finite element discretization of the unit cell with prescribed quasi-periodic Bloch boundary conditions have been developed and applied to model the scattering of acoustic waves in periodic materials and to obtain reliable approximations of the band structure and the identification of band gaps (e.g. Langlet et al., 1995; Hussein and Hulbert, 2006; Hussein, 2009; Huang, 2011, among the others). An alternative approach to the solution of the Floquet–Bloch problem has been proposed by Wu et al. (2004), and Huang and Wu (2011), that

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is based on the plane wave expansion method, i.e. by expanding the elastic moduli, the mass and the unknown displacement vector in Fourier series with respect to the periodicity vectors.

In Nemat-Nasser et al. (2011), the solution is expanded in Fourier series and Bloch waves are obtained by the mixed variational method proposed by Nemat-Nasser (1972). Moreover, the elastodynamics parameters, i.e. the effective elastic moduli and the effective mass density, are obtained as functions of the frequency, which guarantee by construction the exact dispersion relation. A method for the dynamic homogenization of layered elastic composites has been proposed by Nemat-Nasser and Srivastava, 2011, to obtain an accurate evaluation of the averaged overall frequency-dependent dynamic material constitutive relations without the need for a point-wise solution of the field equations; this approach has been extended to three-dimensional periodic elastic composites in Nemat-Nasser and Srivastava (2012).

A different approach to the problem is based on homogenization techniques in which the dynamic macroscopic behavior is derived by upscaling the implied microscopic rules, so obtaining a less detailed description of the dispersive wave propagation, but accurate enough in a selected field of interest. Among these, asymptotic homogenization methods have a great appeal as testified by many contributions (see Guenneau et al., 2012). A homogenization approach devoted to the long wave propagation in periodic media was proposed by Boutin and Auriault (1993), based on an extension of the approach developed by Bensoussan et al. (1978), Sanchez Palencia (1980), and Bakhvalov and Panasenko (1984). In this approach, the asymptotic expansion of the displacement field considers terms up to third order to describe most of the effects of wave scattering. Chen and Fish (2001), Fish et al. (2002), Fish and Chen (2004), have investigated the propagation of waves in periodic bilaminates by developing an asymptotic analysis with multiple spatial and temporal scales and have shown that dispersion effects may be successfully captured by considering fast spatial and slow temporal scales. In addition, a non-local macroscopic equation of motion is inferred from the proposed formulation as a result of the asymptotic homogenization. A formulation in which the slow temporal scale is replaced by a single-frequency time-dependence has been proposed by Vivar-Pérez et al. (2009). Here an asymptotic expansion for the main frequencies is assumed to exist and the resulting treatment appears to be in good agreement with the previously mentioned model by Chen and Fish (2001).

Noting that the solutions resulting by these approaches cannot fully reproduce high-frequency dynamic behaviors characteristic of microstructured periodic materials, such as the presence of band gaps or negative refraction, Craster et al. (2010) proposed a high-frequency asymptotic procedure which is based upon perturbing about standing wave solutions occurring at particular frequencies in the heterogeneous material. Notably, by introducing a dimensionless ratio involving the angular frequency, the microstructure characteristic length and the propagation velocity in a phase, the cell problems take the form of spectral problems. The obtained results accurately reproduce the behavior of elastic waves near the edges of the Brillouin zone, notably the Bloch mode spectrum (see also Antonakakis et al., 2013). Wave propagation in periodic elastic composite having a sufficiently large contrast in material parameters have been analysed through asymptotic approaches by Zhikov (2000), Smyshlyaev (2009), and Auriault and Boutin (2012), among the others. These approaches are based on the observation that in high-contrast elastic composite materials, higher Bloch modes may become part of the low frequency response thus enabling to obtain the band-gap structure as solution of a homogenized spectral problem. Andrianov et al. (2008, 2011), have developed a twofold approach to the analysis of 2D wave propagation. The long-wave approximation valid for the low-frequency range has been obtained through a higher order asymptotic homogenization while for the high-frequency range a Floquet–Bloch analysis based on plane wave expansions has been developed.

Focusing on long-wave propagation, beside the above mentioned models proposed by Fish and co-workers and Andrianov and co-workers, the contribution of Wang and Sun (2002) and Sun and Huang (2007) deserves to be considered, who proposed a continuum model with classical constitutive equations and including enhanced micro-inertia terms in the kinetic energy density to better represent material heterogeneities. Finally, a second-gradient computational homogenization based on a variational-asymptotic approach has been proposed by Bacigalupo and Gambarotta (2012a) based on proper micro-fluctuation functions to represent the material microstructure. This approach provides sufficiently accurate results in case of compressional and shear waves travelling in the direction normal to the layers, but results in a lower accuracy of the dispersive functions in the case of compressional waves along the layers.

In the present paper, a dynamic homogenization procedure for the analysis of wave propagation in periodic materials based on a higher order continuum model is proposed. The aim to obtain a sufficiently accurate approximation of the lowest (acoustic) branch of the dispersion curves obtained by the Floquet–Bloch theory and a non-local homogenized model that may be a useful tool for the wave propagation analysis in bounded domains. A variational-asymptotic homogenization technique recently developed by Bacigalupo and Gambarotta (2012a) based on a combination of the asymptotic and variational methods (Smyshlyaev and Cherednichenko, 2000) is reformulated in the present paper to obtain a second-gradient continuum model able to provide a more accurate description of the dispersion curves. An enhanced description of the multi-scale kinematics is developed through suitable micro-fluctuation functions. These ones are obtained as the solution of appropriate cell problems derived by an asymptotic procedure and are applied to relate the effective inertial and elastic properties with those of the individual phases. By exploiting these solutions, the equation of motion is derived at the macro-scale for the second-gradient continuum through the Hamilton's least action principle. The proposed model is applied to the study of the wave propagation in layered materials with orthotropic phases and axis of orthotropy parallel to the direction of layering, in which case the overall elastic and inertial constants can be determined analytically. To assess the reliability of the proposed procedure, the obtained dispersion functions are compared with those obtained from the solution of the Floquet–Bloch problem and from the non-local model proposed in Bacigalupo and Gambarotta (2012a,b), respectively.

2. Equation of motion in elastic periodic materials

Let us consider a two-dimensional heterogeneous body characterised by a periodic microstructure undergoing to small strains (Fig. 1a); its constituent elements are modelled as an elastic Cauchy continuum. The position vector $\mathbf{x}$ of a material point is denoted by its components (x_1, x_2) with respect to the reference $(0, \mathbf{e}_1, \mathbf{e}_2)$. The periodic materials is fully characterized by the periodic cell, the smallest plane portion that contains all the essential information about the microstructure arrangement. The elementary cell $\mathbf{A} = [0, \varepsilon] \times [0, \delta\varepsilon]$ having microstructural characteristic size ε is shown in Fig. 1b, which is spanned by the two independent orthogonal vectors $\mathbf{v}_1 = d_1\mathbf{e}_1, \mathbf{v}_2 = d_2\mathbf{e}_2$, so that the boundary C of the cell is made up of two pairs of opposite sides corresponding to each other by means of a translation along $\mathbf{v}_1$ or $\mathbf{v}_2$. Accordingly, the elasticity tensor $\mathbb{C}^{m,\varepsilon}(\mathbf{x})$ and the mass density $\rho^{\varepsilon}(\mathbf{x})$ are $\mathbf{A}$-periodic, i.e. $\mathbb{C}^{m,\varepsilon}(\mathbf{x} + \mathbf{v}_i) = \mathbb{C}^{m,\varepsilon}(\mathbf{x})$, $\rho^{\varepsilon}(\mathbf{x} + \mathbf{v}_i) = \rho^{\varepsilon}(\mathbf{x})$, $i = 1,2$, $\forall \mathbf{x} \in \mathbf{A}$. This suggest to consider a unit cell $Q = [0, 1] \times [0, \delta]$ that reproduces

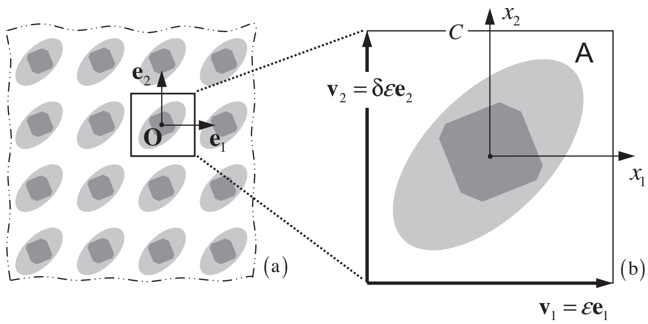

Fig. 1. (a) Heterogeneous material; (b) periodic cell A and periodicity vectors.

the periodic microstructure by rescaling with the characteristic size ε so that the two distinct scales are represented by the macroscopic (slow) variables $\mathbf{x}\in\mathbf{A}$ and the microscopic (fast) variable $\xi=\mathbf{x}/\varepsilon\in Q$. The mapping of both the elasticity tensor and of the mass density may be defined on Q as follows: $\mathbb{C}^{m,\varepsilon}(\mathbf{x})=\mathbb{C}^m(\xi=\mathbf{x}/\varepsilon)$, $\rho^\varepsilon(\mathbf{x})=\rho(\xi=\mathbf{x}/\varepsilon)$, respectively.

The micro-displacement $\mathbf{u}(\mathbf{x},t)$ at time t of a material point of the heterogeneous elastic medium initially located at $\mathbf{x}$ is considered together with the corresponding micro-strain tensor $\varepsilon(\mathbf{x},t)=sym\nabla\mathbf{u}(\mathbf{x},t)$ and the micro-stress tensor $\sigma(\mathbf{x},t)=\mathbb{C}^m(\frac{\mathbf{x}}{\varepsilon})\varepsilon(\mathbf{x},t)$ which has to satisfy the local equation of motion $\operatorname{div}\sigma(\mathbf{x},t)=\rho(\frac{\mathbf{x}}{\varepsilon})\ddot{\mathbf{u}}(\mathbf{x},t)-\mathbf{f}(\mathbf{x},t)$ where $\mathbf{f}(\mathbf{x},t)\in L^2[\mathbb{R}^2\times\mathbb{R}]$ is the body force depending on the slow variable $\mathbf{x}$ and on the time variable t. The resulting set of partial differential equations is written in the form

$$\operatorname{div}\left(\mathbb{C}^m\left(\frac{\mathbf{x}}{\varepsilon}\right)\nabla\mathbf{u}(\mathbf{x},t)\right)=\rho\left(\frac{\mathbf{x}}{\varepsilon}\right)\ddot{\mathbf{u}}(\mathbf{x},t)-\mathbf{f}(\mathbf{x},t) \tag{1}$$

and hence the displacement may be seen in the classical form $\mathbf{u}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon},t)$ as a function of both the slow and the fast variable (the elasticity tensor has the property $\mathbb{C}^m\mathbf{Z}=\mathbb{C}^m sym\mathbf{Z},\ \forall\mathbf{Z}$). By applying to Eq. (1) the Fourier transform respect to the time variable t, i.e. $\mathcal{F}[\mathbf{u}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon},t)]=\int_{-\infty}^{+\infty}\mathbf{u}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon},t)e^{I\omega t}dt=\hat{\mathbf{u}}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon})$ (with ω angular frequency and $I^2=-1$), the local motion equation in the frequency-space is obtained and takes the form

$$\operatorname{div}\left(\mathbb{C}^m\left(\frac{\mathbf{x}}{\varepsilon}\right)\nabla\hat{\mathbf{u}}(\mathbf{x})\right)+\omega^2\rho\left(\frac{\mathbf{x}}{\varepsilon}\right)\hat{\mathbf{u}}(\mathbf{x})=-\hat{\mathbf{f}}(\mathbf{x}). \tag{2}$$

This formulation allows to obtain the solution of the equation of motion by means of an asymptotic expansion of the micro-displacement Fourier transform in the frequency space (and not directly in the physical space) with the advantage of ignoring the time variable t in the asymptotic expansions (compare with Bakhvalov and Panasenko, 1984, p. 144, Eq. (12)). Moreover, Eq. (2) is formally similar to the equation of motion (1) for a time harmonic dependence $\mathbf{u}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon},t)=\mathbf{v}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon})e^{-I\omega t}$ assumed understood (see Boutin and Auriault (1993), Craster et al., 2010).

For vanishing body forces, Eq. (2) results in a plane elastic wave, solution of a cell problem defined on the domain $\mathbf{A}$ with Floquet–Bloch boundary conditions (Nemat-Nasser, 1972). A complementary approach developed in terms of an equivalent homogeneous continuum may be considered for unbounded and bounded domain, which approximates the rigorous solution. In this case, dispersive waves are described from the microstructure data by means of a non-local continuum which is derived through a variational-asymptotic homogenization technique according to Smyshlyaev and Cherednichenko (2000). To this purpose an asymptotic expansion for the micro-displacement is considered in terms of the parameter ε, that keeps the dependence on the slow variable $\mathbf{x}$ separate from the fast one ξ (so that two distinct scales are represented). In Sections 3 and 4 this procedure will be illustrated and the elastic and inertial properties of the equivalent second gradient continuum will be derived in terms of geometrical and mechanical properties of the microstructure. Notably, the perturbation functions obtained by this procedure, embedded in the down-scaling of the frequency Fourier transform of the displacement field, depend (unlike Bacigalupo and Gambarotta, 2012a,b) not only on the fast variable ξ but also on the ratio ω/k (where $k=\|\mathbf{k}\|$, with $\mathbf{k}$ wave-vector) and on the wave vector direction $\mathbf{n}$, as will be shown in the next Section.

3. Multi-scale kinematics and asymptotic solution of the heterogeneous problem

In analogy to the approach by Bakhvalov and Panasenko (1984), the Fourier transform $\hat{\mathbf{u}}(\mathbf{x})$ of the micro-displacement field $\mathbf{u}(\mathbf{x})$ in the frequency-space is represented through the following asymptotic expansion

$$\begin{aligned}\hat{u}_i\left(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon}\right)&=\left(u_i^*(\mathbf{x})+\sum_{l=1}^{+\infty}\varepsilon^l\sum_{|q|=l}\mathrm{N}^l_{ijq}(\xi)\frac{\partial^{|q|}}{\partial\mathbf{x}^q}u_j^*(\mathbf{x})\right)\Bigg|_{\xi=\frac{\mathbf{x}}{\varepsilon}}\\&=\left(u_i^*(\mathbf{x})+\varepsilon\mathrm{N}^1_{ijq_1}(\xi)\frac{\partial u_j^*(\mathbf{x})}{\partial x_{q_1}}+\varepsilon^2\mathrm{N}^2_{ijq_1q_2}(\xi)\frac{\partial^2u_j^*(\mathbf{x})}{\partial x_{q_1}\partial x_{q_2}}+\cdots\right)\Bigg|_{\xi=\frac{\mathbf{x}}{\varepsilon}}\end{aligned} \tag{3}$$

in terms of the parameter ε, that is assumed to be small in comparison to a macroscopic characteristic size, to a wavelength or to a characteristic size related to the fluctuation of body forces. In Eq. (3), q is a multi-index, $u_i^*(\mathbf{x})$ is a function that only depends on the macroscopic coordinate $\mathbf{x}$ and the functions $\mathrm{N}^l_{ijq}(\xi)$ are Q -periodic perturbation functions having vanishing mean value on the unit cell Q, i.e. $\left\langle N^l_{ijq}(\xi)\right\rangle=\frac{1}{\delta}\int_Q N^l_{ijq}(\xi)d\xi=0$. As shown by Bakhvalov and Panasenko (1984) and discussed by Smyshlyaev and Cherednichenko (2000), (Section 2.2), the double-series structure (3) is equivalent to the well-known form separating the *fast* and the *slow* variables assumed in the asymptotic techniques (Bensoussan et al., 1978; Sanchez Palencia, 1980; Boutin and Auriault, 1993).

The Fourier transform $\hat{\mathbf{U}}(\mathbf{x})$ of the macro-displacement field is defined as the average of the Fourier transform of the micro-displacement on the unit cell Q

$$\begin{aligned}\hat{\mathbf{U}}(\mathbf{x},t)&=\left\langle\hat{\mathbf{u}}\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}+\zeta,t\right)\right\rangle=\frac{1}{\delta}\int_Q\hat{\mathbf{u}}(\mathbf{x},\xi+\zeta,t)\Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}}d\xi\\&=\frac{1}{\delta}\int_Q\hat{\mathbf{u}}(\mathbf{x},\xi+\zeta,t)\Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}}d\zeta,\end{aligned}$$

where $\zeta\in Q$ and the vector $\varepsilon\zeta\in\mathbf{A}$ measure all the possible translations of the heterogeneous medium compared to a grid of cells having characteristic size ε (see Smyshlyaev and Cherednichenko, 2000). By Eq. (3), the components of $\hat{\mathbf{U}}(\mathbf{x})$ take the form

$$\hat{U}_i(\mathbf{x})\doteq\left\langle\hat{u}_i\left(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}+\zeta\right)\right\rangle=\frac{1}{\delta}\int_Q\left(u_i^*(\mathbf{x})+\sum_{l=1}^{+\infty}\varepsilon^l\sum_{|p|=l}\mathrm{N}^l_{ijq}(\xi+\zeta)\frac{\partial^{|q|}}{\partial\mathbf{x}^q}u_j^*(\mathbf{x})\right)\Bigg|_{\xi=\frac{\mathbf{x}}{\varepsilon}}d\zeta=u_i^*(\mathbf{x}) \tag{4}$$

because the perturbation functions have zero mean value on Q. The macro-displacement field $\mathbf{U}(\mathbf{x})$ is obtained by the inverse Fourier transform and it results to be $\mathbf{U}(\mathbf{x})=\langle\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}+\zeta)\rangle$.

Substituting the micro-displacement field (3) in Eq. (2), once noted that $\frac{\partial}{\partial x_j}u_i(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon})=\left(\frac{\partial u_i}{\partial x_j}+\frac{1}{\varepsilon}\frac{\partial u_i}{\partial\xi_j}\right)\Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}}=\left(\frac{\partial u_i}{\partial x_j}+\frac{u_{i,j}}{\varepsilon}\right)\Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}}$ (where the comma derivative notation is assumed with respect to the *fast* variable) and by Eq. (4), the equation of motion at the micro-scale may be rewritten as

$$\begin{aligned}&\left\{\frac{1}{\varepsilon}(C^m_{ijhk}N^1_{hpr_1,k}+C^m_{ijpr_1})_{,j}\hat{H}_{pr_1}(\mathbf{x})\right.\\&+\left[\left(C^m_{ijhk}N^2_{hpr_1r_2,k}+\frac{1}{2}(C^m_{ijhr_2}N^1_{hpr_1}+C^m_{ijhr_1}N^1_{hpr_2})\right)_{,j}\right.\\&\left.+\frac{1}{2}\left(C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right)\right]\hat{\kappa}_{pr_1r_2}(\mathbf{x})\\&\left.+\cdots+\omega^2\rho\left(\hat{U}_i+\varepsilon N^1_{ipr_1}\hat{H}_{pr_1}+\cdots\right)\right\}\Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}}=-\hat{f}_i(\mathbf{x}),\quad i=1,2,\end{aligned}\tag{5}$$

$\hat{H}_{ir_1}(\mathbf{x})=\frac{\partial\hat{U}_i}{\partial x_{r_1}}$ being the displacement gradient and $\hat{\kappa}_{ir_1r_2\ldots r_m}(\mathbf{x})=\frac{\partial^m\hat{U}_i}{\partial x_{r_1}\partial x_{r_2}\ldots\partial x_{r_m}}$ the higher-order strain in the frequency-space, respectively.

By applying the Fourier transform to Eq. (5) with respect to the slow variable $\mathbf{x}$, i.e. $\mathcal{F}[\hat{\mathbf{U}}(\mathbf{x})]=\int_{-\infty}^{+\infty}\hat{\mathbf{U}}(\mathbf{x})e^{I\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}=\int_{-\infty}^{+\infty}\hat{\mathbf{U}}(\mathbf{x})e^{Ik_sx_s}d\mathbf{x}=\hat{\hat{\mathbf{U}}}$ (with $\mathbf{k}$ bi-dimensional vector), the local equation of motion in the transformed space takes the form

$$\begin{aligned}&\left\{-\frac{1}{\varepsilon}(C^m_{ijhk}N^1_{hpr_1,k}+C^m_{ijpr_1})_{,j}Ik_{r_1}\hat{\hat{U}}_p\right.\\&-\left[\left(C^m_{ijhk}N^2_{hpr_1r_2,k}+\frac{1}{2}(C^m_{ijhr_2}N^1_{hpr_1}+C^m_{ijhr_1}N^1_{hpr_2})\right)_{,j}\right.\\&\left.+\frac{1}{2}\left(C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right)\right]k_{r_1}k_{r_2}\hat{\hat{U}}_p\\&\left.+\cdots+\omega^2\rho\left(\hat{\hat{U}}_i-\varepsilon N^1_{ipr_1}Ik_{r_1}\hat{\hat{U}}_p+\cdots\right)\right\}\Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}}=-\hat{\hat{f}}_i,\quad i=1,2.\end{aligned}\tag{6}$$

In order to obtain a formulation involving the macroscopic state variables alone, namely a PDE with constant coefficients, the unknown functions N^1_{ipr}, $N^2_{ipr_1r_2}$ and $N^m_{ipr_1\ldots r_m}$ have to fulfil the following non-homogeneous equations (*cell problems*)

$$\begin{aligned}&-I[(C^m_{ijhk}N^1_{hpr_1,k})_{,j}+C^m_{ijpr_1,j}]k_{r_1}=\tilde{h}^1_{ip}\\&-\left[(C^m_{ijhk}N^2_{hpr_1r_2,k})_{,j}+\tfrac{1}{2}\left((C^m_{ijhr_2}N^1_{hpr_1}+C^m_{ijhr_1}N^1_{hpr_2})_{,j}+C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}\right.\right.\\&\qquad\left.\left.+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right)\right]k_{r_1}k_{r_2}+\omega^2\rho\delta_{ip}=\tilde{h}^2_{ip}\\&\qquad\vdots\\&(-I)^m\left\{(C^m_{ijhk}N^m_{hpr_1\ldots r_m,k})_{,j}+\tfrac{1}{m!}\sum_{\wp(r)}\left[(C^m_{ijlhr_m}N^{m-1}_{hpr_1\ldots r_{m-1}})_{,j}+C^m_{ir_mhh}N^{m-1}_{hpr_1\ldots r_{m-1},k}+\right.\right.\\&\qquad\left.\left.+C^m_{ir_mhr_{m-1}}N^{m-2}_{hpr_1\ldots r_{m-2}}\right]\right\}k_{r_1}k_{r_2}\ldots k_{r_m}+(-I)^{m-2}\omega^2\rho N^{m-2}_{ipr_1\ldots r_{m-2}}k_{r_1}k_{r_2}\ldots k_{r_{m-2}}=\tilde{h}^m_{ip},\end{aligned}\tag{7}$$

where the symbol $\wp(r)$ denotes all the possible permutations of the multi-index r, the constant $\tilde{h}^1_{ip}$ is vanishing ($\tilde{h}^1_{ip}=0$) and $\tilde{h}^2_{ip}$, $\tilde{h}^m_{ip}$ are defined as follows

$$\begin{aligned}&\tilde{h}^2_{ip}=-\tfrac{1}{2}\left\langle C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right\rangle k_{r_1}k_{r_2}+\langle\rho\rangle\omega^2\delta_{ip}\\&\qquad\vdots\\&\tilde{h}^m_{ip}=(-I)^m\tfrac{1}{m!}\sum_{\wp(r)}\left\langle C^m_{ir_mhh}N^{m-1}_{hpr_1\ldots r_{m-1},k}+C^m_{ir_mhr_{m-1}}N^{m-2}_{hpr_1\ldots r_{m-2}}\right\rangle k_{r_1}k_{r_2}\ldots k_{r_m}\\&\quad+(-I)^{m-2}\omega^2\left\langle\rho N^{m-2}_{ipr_1\ldots r_{m-2}}\right\rangle k_{r_1}k_{r_2}\ldots k_{r_{m-2}}.\end{aligned}\tag{8}$$

Once written the vector $\mathbf{k}$ in terms of the wave direction vector $\mathbf{n}$, i.e. $\mathbf{k}=k\mathbf{n}$ (with $k=\|\mathbf{k}\|$ and $\|\mathbf{n}\|=1$) and recalling (8), the cell problems (7) take the form

$$\begin{aligned}&-I[(C^m_{ijhk}N^1_{hpr_1,k})_{,j}+C^m_{ijpr_1,j}]kn_{r_1}=0\\&-\left[(C^m_{ijhk}N^2_{hpr_1r_2,k})_{,j}+\frac{1}{2}\left((C^m_{ijhr_2}N^1_{hpr_1}+C^m_{ijhr_1}N^1_{hpr_2})_{,j}+C^m_{ir_1hk}N^1_{hpr_2,k}\right.\right.\\&\quad\left.+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right)\\&\quad\left.-\frac{1}{2}\left\langle C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right\rangle\right]k^2n_{r_1}n_{r_2}\\&\quad+\omega^2(\rho-\langle\rho\rangle)\delta_{ip}\delta_{r_1r_2}n_{r_1}n_{r_2}=0\\&\qquad\vdots\\&(-I)^m\left\{(C^m_{ijhk}N^m_{hpr_1\ldots r_m,k})_{,j}+\frac{1}{m!}\sum_{\wp(r)}\left[(C^m_{ijlhr_m}N^{m-1}_{hpr_1\ldots r_{m-1}})_{,j}\right.\right.\\&\quad+C^m_{ir_mhh}N^{m-1}_{hpr_1\ldots r_{m-1},k}+C^m_{ir_mhr_{m-1}}N^{m-2}_{hpr_1\ldots r_{m-2}}\\&\quad\left.\left.-\left\langle C^m_{ir_mhh}N^{m-1}_{hpr_1\ldots r_{m-1},k}+C^m_{ir_mhr_{m-1}}N^{m-2}_{hpr_1\ldots r_{m-2}}\right\rangle\right]\right\}k^mn_{r_1}n_{r_2}\ldots n_{r_m}\\&\quad+(-I)^{m-2}\omega^2k^{m-2}\frac{1}{m!}\sum_{\wp(r)}\left[\left(\rho N^{m-2}_{ipr_1\ldots r_{m-2}}-\left\langle\rho N^{m-2}_{ipr_1\ldots r_{m-2}}\right\rangle\right)\delta_{r_mr_{m-1}}\right]\\&\quad\times n_{r_1}n_{r_2}\ldots n_{r_m}=0.\end{aligned}\tag{9}$$

Due to the structure of these equations, the perturbation functions are assumed in the form $N^1_{ipr}=N^{1\_S}_{ipr}$, $N^2_{ipqr}=N^{2\_S}_{ipqr}+c^2N^{2\_D}_{ipqr}$, $N^m_{ipr_1\ldots r_m}=N^{m\_S}_{ipr_1\ldots r_m}+c^2N^{m\_D}_{ipr_1\ldots r_m}=N^{m\_S}_{ipr_1\ldots r_m}+c^2\sum_{j=1}^{m-2}c^{2(j-1)}N^{m\_D_j}_{ipr_1\ldots r_m}$, $m\geqslant3$ (with $c=\omega/k$ where $k=\|\mathbf{k}\|$) as the superposition of the contributions associated with the elastic moduli (*static terms*) and those ones associated with both the elastic moduli and the mass density (*dynamic terms*). The following cell problems expressed in terms of the functions $N^{1\_S}_{ipr}$, $N^{2\_S}_{ipqr}$, $N^{2\_D}_{ipqr}$, $N^{m\_S}_{ipq_1\ldots q_m}$, $N^{m\_D_j}_{ipq_1\ldots q_m}$ are obtained

$$\begin{aligned}&(C^m_{ijhk}N^1_{hpr_1,k})_{,j}=-C^m_{ijpr_1,j}\\&(C^m_{ijhk}N^{2\_S}_{hpr_1r_2,k})_{,j}=-\frac{1}{2}\left[\left(C^m_{ijhr_2}N^1_{hpr_1}+C^m_{ijhr_1}N^1_{hpr_2}\right)_{,j}+C^m_{ir_1hk}N^1_{hpr_2,k}\right.\\&\qquad\left.+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right]\\&\qquad+\frac{1}{2}\left\langle C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right\rangle\\&(C^m_{ijhk}N^{2\_D}_{hpr_1r_2,k})_{,j}=(\rho-\langle\rho\rangle)\delta_{ip}\delta_{r_1r_2}\\&\qquad\vdots\\&(C^m_{ijhk}N^{m\_S}_{hpr_1\ldots r_m,k})_{,j}=-\frac{1}{m!}\sum_{\wp(q)}\left[(C^m_{ijlhr_m}N^{m-1\_S}_{hpr_1\ldots r_{m-1}})_{,j}+C^m_{ir_mhh}N^{m-1\_S}_{hpr_1\ldots r_{m-1},k}\right.\\&\qquad\left.+C^m_{ir_mhr_{m-1}}N^{m-2\_S}_{hpr_1\ldots r_{m-2}}\left\langle C^m_{ir_mhh}N^{m-1\_S}_{hpr_1\ldots r_{m-1},k}+C^m_{ir_mhr_{m-1}}N^{m-2\_S}_{hpr_1\ldots r_{m-2}}\right\rangle\right]\\&(C^m_{ijhk}N^{m\_D}_{hpr_1\ldots r_m,k})_{,j}=-\frac{1}{m!}\sum_{\wp(r)}\left[\left(C^m_{ijlhr_m}N^{m-1\_D}_{hpr_1\ldots r_{m-1}}\right)_{,j}+C^m_{ir_mhh}N^{m-1\_D}_{hpr_1\ldots r_{m-1},k}+C^m_{ir_mhr_{m-1}}N^{m-2\_D}_{hpr_1\ldots r_{m-2}}\right.\\&\qquad-\left\langle C^m_{ir_mhh}N^{m-1\_D}_{hpr_1\ldots r_{m-1},k}+C^m_{ir_mhr_{m-1}}N^{m-2\_D}_{hpr_1\ldots r_{m-2}}\right\rangle\\&\qquad+c^2\left(\rho N^{m-2}_{ipr_1\ldots r_{m-2}}-\left\langle\rho N^{m-2}_{ipr_1\ldots r_{m-2}}\right\rangle\right)\delta_{r_mr_{m-1}}.\end{aligned}\tag{10}$$

From the previous definition $N^{m\_D}_{hpr_1\ldots r_m}=\sum_{j=1}^{m-2}c^{2(j-1)}N^{m\_D_j}_{ipr_1\ldots r_m}$ and by considering each order of $c^{2(j-1)}$, the last equation in (10) may be decomposed in $(m-2)$ PDEs in terms of the $(m-2)$ perturbation functions $N^{m\_D_j}_{ipq_1\ldots q_m}$. It follows that the differential cell problems obtained from (10) are independent of $c=\omega/k$. In the case of homogeneous mass density, the *dynamic terms* vanish and standard cell problems are obtained (Bakhvalov and Panasenko, 1984; Boutin and Auriault, 1993; Boutin, 1996; Smyshlyaev and Cherednichenko, 2000). Conversely, in the case of heterogeneous mass density, additional cell problems are obtained whose solutions are here denoted as *dynamic terms*. To sum up, two sets of uncoupled cell problems are obtained from the proposed procedure. The cell problems of the first set depend exclusively on the elastic moduli of the phases and their solutions are denoted as *static terms*. The cell problems of the second set also depend on the mass density and provide the *dynamic terms*. It is worth to note that the cell problems obtained by superimposing the two sets of cell problems are similar, at the different orders of ε, to those formulated through different procedures by Boutin and Auriault (1993) and Vivar-Pérez et al., 2009, if, in the present formulation, the constant c is assumed as the wave velocity $\hat{c}$ in the classical homogenized continuum (see Section 4).

From (7)–(10), the averaged equations of infinite order in the transformed space take the form

$$k^2(h^2_{ip}+\cdots+\varepsilon^{m-2}h^m_{ip}+\cdots)\hat{\hat{U}}_p=\hat{\hat{f}}_i,\quad i=1,2,\tag{11}$$

where the constants h^2_{ip}, h^m_{ip} are expressed in terms of the perturbation functions $N^{1\_S}_{ipr}$, $N^{2\_S}_{ipqr}$, $N^{2\_D}_{ipqr}$, $N^{m\_S}_{ipq_1\ldots q_m}$, $N^{m\_D_j}_{ipq_1\ldots q_m}$ and of the parameters k, c, as follows

$$\begin{aligned}
h^2_{ip}&=\tfrac{1}{2}\left\langle C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right\rangle n_{r_1}n_{r_2}-\langle\rho\rangle c^2\delta_{ip}\\
&\vdots\\
h^m_{ip}&=-(-I)^m\tfrac{1}{m!}\sum_{\wp(r)}\left\langle C^m_{ir_mhh}N^{m-1}_{hpr_1\ldots r_{m-1},k}+C^m_{ir_mhr_{m-1}}N^{m-2}_{hpr_1\ldots r_{m-2}}\right\rangle k^{m-2}n_{r_1}n_{r_2}\ldots n_{r_m}\\
&\quad-(-I)^{m-2}c^2\left\langle\rho N^{m-2}_{ipr_1\ldots r_{m-2}}\right\rangle k^{m-2}n_{r_1}n_{r_2}\ldots n_{r_{m-2}}.
\end{aligned}\tag{12}$$

By the algebraic system (11) the components $\hat{\hat{U}}_p$ of Fourier transform of the macro-displacement vector may be obtained in the transformed space and then, by the inverse Fourier transform the macro-displacement field is determined $U_p=\mathcal{F}^{-1}[\mathcal{F}^{-1}[\hat{\hat{U}}_p]]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\hat{\hat{U}}_pe^{-I\mathbf{k}\cdot\mathbf{x}}d\mathbf{k})e^{-I\omega t}d\omega$ $=\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{+\infty}\hat{\hat{U}}_pe^{-Ik\mathbf{n}\cdot\mathbf{x}}kdkd\vartheta)e^{-I\omega t}d\omega$, with $n_1=\cos\vartheta$ and $n_2=\sin\vartheta$, respectively.

It is well known that the truncation of Eq. (11) at a suitable order to obtain higher order field equations generally may lead to problems because the symmetry of the higher-order elastic moduli associated to constants h^2_{ip}, h^m_{ip} is not generally guaranteed (a discussion of these problems is given in Smyshlyaev and Cherednichenko, 2000). A possibility to circumvent these drawbacks is to introduce an asymptotic expansion of the macro-displacement in the averaged equations of infinite order thus obtaining a sequence of standard dynamic problems with non-zero body forces (equivalent approaches are derived in Boutin and Auriault (1993) and Vivar-Pérez et al. (2009)). Alternatively, multipolar models of equivalent continuous may be obtained through homogenization techniques based on a variational formulation in conjunction with asymptotic considerations (see Smyshlyaev and Cherednichenko, 2000).

From the point of view of applications, to obtain higher order field equations by the asymptotic approach (through a truncation of the average equation of infinite order) a number of perturbation functions greater than those required by a variational-asymptotic approach is required. In fact, to obtain the equation of motion of a second gradient non-local continuum (where the fourth derivatives in the slow variable are involved being all the terms up to ε^2 considered), is necessary to determine the perturbation functions N^1_{ipr}, N^2_{ipqr}, $N^3_{ipq_1q_2q_3}$ (as can be inferred in Bensoussan et al., 1978; Bakhvalov and Panasenko, 1984; Smyshlyaev and Cherednichenko, 2000), while to apply the variational-asymptotic technique the functions, N^2_{ipqr} are sufficient (see Bacigalupo and Gambarotta, 2012a,b).

In agreement with these observations the equation of motion of a homogeneous second gradient non-local continuum are derived in the next Section through an extension of the variational-asymptotic procedure recently proposed by Bacigalupo and Gambarotta (2012a).

4. Second gradient homogenized equations

The displacement Fourier transform in the frequency-space may be approximated by the truncated second order asymptotic series (with reference to (3)) as follows

$$\begin{aligned}
\hat{\mathbf{u}}\left(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon}\right)\approx\hat{\mathbf{u}}^{II}\left(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon}\right)&=\hat{\mathbf{U}}(\mathbf{x})+\varepsilon\mathbf{N}^1(\xi)\\
&:\hat{\mathbf{H}}(\mathbf{x})+\varepsilon^2\mathbf{N}^2(\xi)\vdots\hat{\boldsymbol{\kappa}}(\mathbf{x}).
\end{aligned}\tag{13}$$

The micro-fluctuation functions $\mathbf{N}^1(\xi)$ and $\mathbf{N}^2(\xi)$ (with components $N^1_{ipr}=N^{1\_S}_{ipr}$ and $N^2_{ipqr}=N^{2\_S}_{ipqr}+c^2N^{2\_D}_{ipqr}$ with $c=\omega/k$) are the solution of the *cell problem* (10.1), (10.2), (10.3); these functions depend on the fast coordinate and are Q-periodic with zero mean over Q, namely $\left\langle N^1_{ikl}(\xi)\right\rangle=0$ and $\left\langle N^2_{iklp}(\xi)\right\rangle=0$.

To define the perturbation functions, the constant c may be determined by solving Eqs. (11) only considering the zeroth order terms. For vanishing body forces (homogeneous problem - wave propagation) the homogeneous system $h^2_{ip}\hat{\hat{U}}_p=0$ (with $i=1,2$) is obtained that takes the form of an eigenvalue problem

$$\begin{bmatrix}Q^2_{11}-\langle\rho\rangle c^2 & Q^2_{12}\\ Q^2_{21} & Q^2_{22}-\langle\rho\rangle c^2\end{bmatrix}\begin{Bmatrix}\hat{\hat{U}}_1\\ \hat{\hat{U}}_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}\tag{14}$$

$Q^2_{ip}=H^2_{ipr_1r_2}n_{r_1}n_{r_2}$ being the component of the acoustical first order tensor, $H^2_{ipr_1r_2}=\frac{1}{2}\left\langle C^m_{ir_1hk}N^1_{hpr_2,k}+C^m_{ir_1pr_2}+C^m_{ir_2hk}N^1_{hpr_1,k}+C^m_{ir_2pr_1}\right\rangle$ the classical overall elastic fourth order tensor ($H^2_{ipr_1r_2}=C_{ipr_1r_2}$ in Eq. (20)) and c the eigenvalue having the meaning of wave phase velocity. For any wave direction vector $\mathbf{n}$, the eigenvalues c^2_ς ($\varsigma=1,2$) of (14) result to be related to the wave velocity $\hat{c}_\varsigma$ in the first order homogenized continuum; the corresponding eigenvectors $\hat{\hat{\mathbf{U}}}^\varsigma$ identify the directions of polarization as shown in Boutin and Auriault (1993). In case of non-zero body forces, the inhomogeneous system (11) truncated to the first order in ε takes the form $k^2h^2_{ip}\hat{\hat{U}}_p=\hat{\hat{f}}_i$ and its solution is obtained as a linear combination of the eigenvectors $\hat{\hat{\mathbf{U}}}^\varsigma$, i.e. $\hat{\hat{U}}_p=A_\varsigma\hat{\hat{U}}^\varsigma_p$, where the constant A_ς is obtained by exploiting the orthogonality properties of the eigenvectors $\hat{\hat{\mathbf{U}}}^\varsigma$ and is expressed in the form $A_\varsigma=\hat{\hat{f}}_i\hat{\hat{U}}^\varsigma_p/[(k^2H^2_{ipr_1r_2}n_{r_1}n_{r_2}-k^2\langle\rho\rangle c^2\delta_{ip})\hat{\hat{U}}^\varsigma_p\hat{\hat{U}}^\varsigma_i]$ (ς not summed). By the inverse Fourier transform applied to (13), the displacement $\mathbf{u}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon},t)$ may be approximated in the form

$$\begin{aligned}
\mathbf{u}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon},t)\approx\mathbf{u}^{II}(\mathbf{x},\xi=\frac{\mathbf{x}}{\varepsilon},t)&=\mathbf{U}(\mathbf{x},t)+\varepsilon\mathbf{N}^1(\xi):\mathbf{H}(\mathbf{x},t)\\
&+\varepsilon^2\mathbf{N}^2(\xi)\vdots\boldsymbol{\kappa}(\mathbf{x},t),
\end{aligned}\tag{15}$$

where $\mathbf{U}(\mathbf{x})$ is the macro-displacement of the second gradient continuum, $\mathbf{H}(\mathbf{x})=H_{ij}\mathbf{e}_i\otimes\mathbf{e}_j=\nabla\mathbf{U}(\mathbf{x})=\frac{\partial U_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j$ is the displacement gradient and $\boldsymbol{\kappa}(\mathbf{x})=\kappa_{ijk}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k=\nabla\otimes\nabla\mathbf{U}(\mathbf{x})=\frac{\partial^2U_i}{\partial x_j\partial x_k}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k$ the second-order strain in the reference ($\mathbf{e}_i, i=1,2$), respectively.

The second gradient homogenized equations are derived via a combination of variational and asymptotic techniques to obtain a weak form of Eq. (1). The variational problem corresponding to Eq. (1) is obtained by the Hamilton's principle

$$\begin{aligned}
\min_{\mathbf{u}(\mathbf{x},t)}\int_{t_0}^{t_1}\mathcal{L}(\mathbf{u})dt&=\min_{\mathbf{u}(\mathbf{x},t)}\int_{t_0}^{t_1}(\mathrm{T}(\mathbf{u})-\Pi(\mathbf{u}))dt\\
&=\min_{\mathbf{u}(\mathbf{x},t)}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\left(\frac{1}{2}\rho\dot{u}_i\dot{u}_i-\frac{1}{2}\frac{\partial u_i}{\partial x_j}C^m_{ijhk}\frac{\partial u_h}{\partial x_k}+f_iu_i\right)d\mathbf{x}dt
\end{aligned}\tag{16}$$

where $\mathcal{L}$ is the Lagrangian functional, T is the kinetic and Π the potential energy functional, respectively, which depend on the micro-displacement $\mathbf{u}(\mathbf{x}, \xi = \frac{\mathbf{x}}{\varepsilon}, t) \in L^2[\mathbb{R}^2 \times \mathbb{R}]$. According to Bacigalupo and Gambarotta (2012a), the averaged Lagrangian functional with respect to the parameter $\zeta \in Q$ is introduced in the form

$$
\begin{aligned}
\bar{\mathcal{L}}(\mathbf{u}) = \langle \mathcal{L}(\mathbf{u}) \rangle &= \frac{1}{\delta} \int_Q \mathcal{L}\left(\mathbf{u}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon} + \zeta, t\right)\right) d\zeta \\
&= \frac{1}{\delta} \int_Q \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho \dot{u}_i \dot{u}_i - \frac{1}{2} \frac{\partial u_i}{\partial x_j} C^m_{ijhk} \frac{\partial u_h}{\partial x_k} + f_i u_i \right) d\mathbf{x} d\zeta \\
&= \frac{1}{\delta} \int_{\mathbb{R}^2} \int_Q \left(\frac{1}{2} \rho \dot{u}_i \dot{u}_i - \frac{1}{2} \frac{\partial u_i}{\partial x_j} C^m_{ijhk} \frac{\partial u_h}{\partial x_k} + f_i u_i \right) d\zeta d\mathbf{x}. \qquad (17)
\end{aligned}
$$

which is based on the argument that the precise "phase" of the microstructure with respect to the body force is generally unknown and a family of translated microstructures should therefore be considered (see also Smyshlyaev and Cherednichenko, 2000).

The second gradient homogenization problem may be now formulated by applying the Hamilton's principle to the averaged Lagrangian functional. If the displacement is restricted to the class of functions $\mathbf{u}^{II}$ having the form (15), the averaged functional depends on the macro-displacement $\mathbf{U}(\mathbf{x}, t)$ and the minimization problem is written in the form

$$
\min_{\mathbf{U}(\mathbf{x},t)} \int_{t_0}^{t_1} \bar{\mathcal{L}}(\mathbf{u}^{II}(\mathbf{U})) dt = \min_{\mathbf{U}(\mathbf{x},t)} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \left\langle \frac{1}{2} \rho \dot{u}^{II}_i \dot{u}^{II}_i - \frac{1}{2} \frac{\partial u^{II}_i}{\partial x_j} C^m_{ijhk} \frac{\partial u^{II}_h}{\partial x_k} + f_i u^{II}_i \right\rangle d\mathbf{x} dt \qquad (18)
$$

The Euler–Lagrangian equation associated to the variational problem (18) is truncated at the third order in the static part and to the fifth order in the inertial part. This approach is in analogy with the idea proposed in Wang and Sun, 2002 and Sun and Huang, 2007, where higher order terms were considered to obtain an accurate description of the micro-inertia effects while the constitutive parameters were assumed within the classical (first-order) description. Accordingly, in the present approach the considered inertial terms are those resulting from the micro-accelerations associated with the second-order strain $\kappa(\mathbf{x}, t)$. The resulting Euler–Lagrangian differential equation takes the form

$$
\begin{aligned}
&\varepsilon^2 h_{ijkpqr} U_{i,jkqr} + \frac{\varepsilon}{2}(h_{ijpqr} - h_{pqijr}) U_{i,jqr} - h_{ijpq} U_{i,jq} \\
&\quad = f_p(\mathbf{x}, t) - \rho_M \ddot{U}_p + \varepsilon \rho_M I_{ipj} \ddot{U}_{i,j} + \varepsilon^2 \rho_M I_{iqpj} \ddot{U}_{i,qj} \\
&\qquad + \varepsilon^3 \rho_M I_{iqpjs} \ddot{U}_{i,qjs} - \varepsilon^4 \rho_M I_{iqjpsr} \ddot{U}_{i,qjsr}, \qquad (19)
\end{aligned}
$$

where $h_{ijpq} = C_{ijpq}$, $h_{ijpqr} = \varepsilon^{-1} Y_{ijpqr}$, $h_{ijkpqr} = \varepsilon^{-2} S_{ijkpqr}$ and ρ_M, I_{ipj}, I_{iqpj}, I_{iqpjs}, I_{iqjpsr} are the effective elastic moduli and the effective inertia parameters of the second-gradient continuum, respectively and the comma derivative notation is referred to the slow variable. Notably, the constitutive homogenization is in agreement with Smyshlyaev and Cherednichenko (2000), and with the technique proposed in Bacigalupo (submitted), and the overall elastic moduli take the form

$$
\begin{aligned}
&C_{pqrs} = \left\langle C^m_{ijkl} B^H_{ijpq} B^H_{klrs} \right\rangle, \quad \frac{Y_{pqrst}}{\varepsilon} = \left\langle C^m_{ijkl} B^H_{ijpq} B^{\kappa\_S}_{klrst} \right\rangle + \hat{c}^2_\varsigma \left\langle C^m_{ijkl} B^H_{ijpq} B^{\kappa\_D}_{klrst} \right\rangle, \\
&\frac{S_{pqhrst}}{\varepsilon^2} = \left\langle C^m_{ijkl} B^{\kappa\_S}_{klrst} B^{\kappa\_S}_{ijpqh} \right\rangle + \hat{c}^2_\varsigma \left\langle C^m_{ijkl} B^{\kappa\_S}_{klrst} B^{\kappa\_D}_{ijpqh} + C^m_{ijkl} B^{\kappa\_D}_{klrst} B^{\kappa\_S}_{ijpqh} \right\rangle \\
&\qquad + \hat{c}^4_\varsigma \left\langle C^m_{ijkl} B^{\kappa\_D}_{klrst} B^{\kappa\_D}_{ijpqh} \right\rangle - \frac{\left\langle A^{H\_\kappa\_S}_{pqhrst} \right\rangle + \hat{c}^2_\varsigma \left\langle A^{H\_\kappa\_D}_{pqhrst} \right\rangle}{12}, \qquad (20)
\end{aligned}
$$

where the localization tensors B^H_{ijpq}, $B^{\kappa\_\beta}_{ijpqr}$ and $A^{H\_\kappa\_\beta}_{pqrjkh}$ (with $\beta = S, D$) are written as follows:

$$
\begin{aligned}
&B^H_{ijpq}\left(\xi = \frac{\mathbf{x}}{\varepsilon}\right) = \frac{1}{4}\left(\delta_{ip}\delta_{jq} + \mathrm{N}^1_{ipq,j} + \delta_{jp}\delta_{iq} + \mathrm{N}^1_{jpq,i} + \delta_{iq}\delta_{jp} + \mathrm{N}^1_{iqp,j} \right. \\
&\qquad \left. + \delta_{jq}\delta_{ip} + \mathrm{N}^1_{jqp,i}\right), \\
&B^{\kappa\_S}_{ijpqr}\left(\xi = \frac{\mathbf{x}}{\varepsilon}\right) = \frac{1}{4}\left(\mathrm{N}^1_{ipq}\delta_{jr} + \mathrm{N}^1_{ipr}\delta_{qj} + 2\mathrm{N}^{2\_\beta}_{ipqr,j} + \mathrm{N}^1_{jpq}\delta_{ir} + \mathrm{N}^1_{jpr}\delta_{qi} + 2\mathrm{N}^{2\_\beta}_{jpqr,i}\right), \\
&B^{\kappa\_D}_{ijpqr}\left(\xi = \frac{\mathbf{x}}{\varepsilon}\right) = \frac{1}{2}\left(\mathrm{N}^{2\_\beta}_{ipqr,j} + \mathrm{N}^{2\_\beta}_{jpqr,i}\right), \\
&A^{H\_\kappa\_\beta}_{pqrjhk}\left(\xi = \frac{\mathbf{x}}{\varepsilon}\right) = \left(C^m_{jkih}\mathrm{N}^{2\_\beta}_{ipqr} + C^m_{jkir}\mathrm{N}^{2\_\beta}_{ipqh} + C^m_{jkiq}\mathrm{N}^{2\_\beta}_{iprh} + C^m_{jhik}\mathrm{N}^{2\_\beta}_{ipqr}\right. \\
&\qquad + C^m_{jhir}\mathrm{N}^{2\_\beta}_{ipqk} + C^m_{jhiq}\mathrm{N}^{2\_\beta}_{iprk} + C^m_{pqir}\mathrm{N}^{2\_\beta}_{ijkh} + C^m_{pqih}\mathrm{N}^{2\_\beta}_{ijkr} \\
&\qquad \left. + C^m_{pqik}\mathrm{N}^{2\_\beta}_{ijhr} + C^m_{priq}\mathrm{N}^{2\_\beta}_{ijkh} + C^m_{prih}\mathrm{N}^{2\_\beta}_{ijkq} + C^m_{prik}\mathrm{N}^{2\_\beta}_{ijhq}\right) \\
&\qquad + \frac{1}{2}\left(C^m_{stir}\mathrm{N}^{2\_\beta}_{ijkh}\mathrm{N}^1_{spq,t} + C^m_{srit}\mathrm{N}^{2\_\beta}_{sjkh}\mathrm{N}^1_{ipq,t} + C^m_{shit}\mathrm{N}^{2\_\beta}_{sjkr}\mathrm{N}^1_{ipq,t}\right. \\
&\qquad + C^m_{stik}\mathrm{N}^{2\_\beta}_{ijhr}\mathrm{N}^1_{spq,t} + C^m_{skit}\mathrm{N}^{2\_\beta}_{sjhr}\mathrm{N}^1_{ipq,t} + C^m_{stiq}\mathrm{N}^{2\_\beta}_{ijkh}\mathrm{N}^1_{spr,t} \\
&\qquad + C^m_{sqit}\mathrm{N}^{2\_\beta}_{sjkh}\mathrm{N}^1_{ipr,t} + C^m_{stih}\mathrm{N}^{2\_\beta}_{ijkq}\mathrm{N}^1_{spr,t} + C^m_{shit}\mathrm{N}^{2\_\beta}_{sjkq}\mathrm{N}^1_{ipr,t} \\
&\qquad + C^m_{stik}\mathrm{N}^{2\_\beta}_{ijhq}\mathrm{N}^1_{spr,t} + C^m_{skit}\mathrm{N}^{2\_\beta}_{sjhq}\mathrm{N}^1_{ipr,t} + C^m_{stih}\mathrm{N}^{2\_\beta}_{ipqr}\mathrm{N}^1_{sjk,t} \\
&\qquad + C^m_{shit}\mathrm{N}^{2\_\beta}_{spqr}\mathrm{N}^1_{ijk,t} + C^m_{stir}\mathrm{N}^{2\_\beta}_{ipqh}\mathrm{N}^1_{sjk,t} + C^m_{srit}\mathrm{N}^{2\_\beta}_{spqh}\mathrm{N}^1_{ijk,t} \\
&\qquad + C^m_{stiq}\mathrm{N}^{2\_\beta}_{iprh}\mathrm{N}^1_{sjk,t} + C^m_{sqit}\mathrm{N}^{2\_\beta}_{sprh}\mathrm{N}^1_{ijk,t} + C^m_{stik}\mathrm{N}^{2\_\beta}_{ipqr}\mathrm{N}^1_{sjh,t} \\
&\qquad + C^m_{skit}\mathrm{N}^{2\_\beta}_{spqr}\mathrm{N}^1_{ijh,t} + C^m_{stir}\mathrm{N}^{2\_\beta}_{ipqk}\mathrm{N}^1_{sjh,t} + C^m_{srit}\mathrm{N}^{2\_\beta}_{spqk}\mathrm{N}^1_{ijh,t} \\
&\qquad \left. + C^m_{stiq}\mathrm{N}^{2\_\beta}_{iprk}\mathrm{N}^1_{sjh,t} + C^m_{sqit}\mathrm{N}^{2\_\beta}_{sprk}\mathrm{N}^1_{ijh,t} + C^m_{stih}\mathrm{N}^{2\_\beta}_{ijkr}\mathrm{N}^1_{spq,t}\right). \qquad (21)
\end{aligned}
$$

Moreover, the effective inertia parameters in Eq. (19) result in the form

$$
\begin{aligned}
&\rho_M = \langle \rho \rangle, I_{ipj} = \frac{\left\langle \rho\left(N^1_{ipj} - N^1_{pij}\right)\right\rangle}{\rho_M}, \\
&I_{iqpj} = \frac{\left\langle \rho\left(\frac{1}{2}(N^1_{rpj}N^1_{riq} + N^1_{rij}N^1_{rpq}) - N^{2\_S}_{ipqj} - N^{2\_S}_{piqj}\right)\right\rangle - \hat{c}^2_\varsigma\left\langle \rho\left(N^{2\_D}_{ipqj} + N^{2\_D}_{piqj}\right)\right\rangle}{\rho_M}, \\
&I_{iqpjs} = \frac{\left\langle \rho\left(N^1_{kip}N^{2\_S}_{kqjs} - N^1_{kqp}N^{2\_S}_{kijs} + 2N^1_{kij}N^{2\_S}_{kqps} - 2N^1_{kqj}N^{2\_S}_{kips} + N^1_{kis}N^{2\_S}_{kqjp} - N^1_{kqs}N^{2\_S}_{kijp}\right)\right\rangle}{4\rho_M} \\
&\qquad + \frac{\hat{c}^2_\varsigma\left\langle \rho\left(N^1_{kip}N^{2\_D}_{kqjs} - N^1_{kqp}N^{2\_D}_{kijs} + 2N^1_{kij}N^{2\_D}_{kqps} - 2N^1_{kqj}N^{2\_D}_{kips} + N^1_{kis}N^{2\_D}_{kqjp} - N^1_{kqs}N^{2\_D}_{kijp}\right)\right\rangle}{4\rho_M}, \\
&I_{ijqpsr} = \left\{\left\langle \rho\left(N^{2\_S}_{kijq}N^{2\_S}_{kpsr} + N^{2\_S}_{kpjq}N^{2\_S}_{kisr} + N^{2\_S}_{kisq}N^{2\_S}_{kpjr} + N^{2\_S}_{kpsq}N^{2\_S}_{kijr} + N^{2\_S}_{kijs}N^{2\_S}_{kpqr} + N^{2\_S}_{kpjs}N^{2\_S}_{kiqr}\right.\right.\right. \\
&\qquad \left.\left. + N^{2\_S}_{kijr}N^{2\_S}_{kpsq} + N^{2\_S}_{kirs}N^{2\_S}_{kpjq}\right)\right\rangle + \hat{c}^4_\varsigma\left\langle \rho\left(N^{2\_D}_{kijq}N^{2\_D}_{kpsr} + N^{2\_D}_{kpjq}N^{2\_D}_{kisr} + N^{2\_D}_{kisq}N^{2\_D}_{kpjr}\right.\right. \\
&\qquad \left.\left. + N^{2\_D}_{kpsq}N^{2\_D}_{kijr} + N^{2\_D}_{kijs}N^{2\_D}_{kpqr} + N^{2\_D}_{kpjs}N^{2\_D}_{kiqr} + N^{2\_D}_{kijr}N^{2\_D}_{kpsq} + N^{2\_D}_{kirs}N^{2\_D}_{kpjq}\right)\right\rangle \\
&\qquad + \hat{c}^2_\varsigma\left\langle \rho\left[\left(N^{2\_S}_{kijq}N^{2\_D}_{kpsr} + N^{2\_S}_{kpjq}N^{2\_D}_{kisr} + N^{2\_S}_{kisq}N^{2\_D}_{kpjr} + N^{2\_S}_{kpsq}N^{2\_D}_{kijr} + N^{2\_S}_{kijs}N^{2\_D}_{kpqr}\right.\right.\right. \\
&\qquad \left. + N^{2\_S}_{kpjs}N^{2\_D}_{kiqr} + N^{2\_S}_{kijr}N^{2\_D}_{kpsq} + N^{2\_S}_{kirs}N^{2\_D}_{kpjq}\right) + \left(N^{2\_D}_{kijq}N^{2\_S}_{kpsr} + N^{2\_D}_{kpjq}N^{2\_S}_{kisr} + N^{2\_D}_{kisq}N^{2\_S}_{kpjr}\right. \\
&\qquad \left.\left.\left.\left. + N^{2\_D}_{kpsq}N^{2\_S}_{kijr} + N^{2\_D}_{kijs}N^{2\_S}_{kpqr} + N^{2\_D}_{kpjs}N^{2\_S}_{kiqr} + N^{2\_D}_{kijr}N^{2\_S}_{kpsq} + N^{2\_D}_{kirs}N^{2\_S}_{kpjq}\right)\right]\right\rangle\right\}/8\rho_M \qquad (22)
\end{aligned}
$$

In case of centro-symmetric periodic cell one obtains $I_{ipj} = 0$, $I_{iqpjs} = 0$. If the mass density is homogeneous in the periodic cell the perturbation functions $N^{2\_D}_{kijq}$ are vanishing, i.e. $N^{2\_D}_{kijq} = 0$, and the overall elastic moduli are independent on the mass density

$$
\begin{aligned}
&C_{pqrs} = \left\langle C^m_{ijkl} B^H_{ijpq} B^H_{klrs} \right\rangle, \quad \frac{Y_{pqrst}}{\varepsilon} = \left\langle C^m_{ijkl} B^H_{ijpq} B^{\kappa\_S}_{klrst} \right\rangle, \\
&\frac{S_{pqhrst}}{\varepsilon^2} = \left\langle C^m_{ijkl} B^{\kappa\_S}_{klrst} B^{\kappa\_S}_{ijpqh} \right\rangle - \frac{\left\langle A^{H\_\kappa\_S}_{pqhrst} \right\rangle}{12}, \qquad (23)
\end{aligned}
$$

where the localization tensors B^H_{ijpq}, $B^{\kappa\_S}_{ijpqr}$ and $A^{H\_\kappa\_S}_{pqrjkh}$ are given in (21) and the inertial terms take the form

$$
\begin{aligned}
&\rho_M = \rho, \quad I_{iqpj} = \frac{1}{2}\left\langle N^1_{rpj}N^1_{riq} + N^1_{rij}N^1_{rpq} \right\rangle, \\
&I_{iqjpsr} = \frac{1}{8}\left\langle N^{2\_S}_{kipq}N^{2\_S}_{kjsr} + N^{2\_S}_{kjpq}N^{2\_S}_{kisr} + N^{2\_S}_{kisq}N^{2\_S}_{kjpr} + N^{2\_S}_{kjsq}N^{2\_S}_{kipr} + N^{2\_S}_{kips}N^{2\_S}_{kjqr}\right. \\
&\qquad \left. + N^{2\_S}_{kjps}N^{2\_S}_{kiqr} + N^{2\_S}_{kipr}N^{2\_S}_{kjsq} + N^{2\_S}_{kirs}N^{2\_S}_{kjpq}\right\rangle. \qquad (24)
\end{aligned}
$$

It is worth to note that the overall elastic moduli and the inertial terms given by Eqs. (23) and (24) coincide with those ones obtained in (Bacigalupo and Gambarotta, 2012b). Moreover, if the third and fourth order inertial terms are neglected, the equation of motion (19) takes the form already obtained in Bacigalupo and Gambarotta (2012a).

Finally, if the material is homogeneous in the periodic cell, i.e. the microstructure disappears, the functions $N^1_{ikl}(\xi)$ and $N^2_{iklp}(\xi)$ are zero and both the elastic moduli and the inertial parameters defined by Eqs. (20), (21), (22), (24) vanish and the equation of motion of the classical continuum is obtained.

By applying the Fourier transform to the equation of motion (19) with respect to the time variable t and to the slow variable $\mathbf{x}$, the algebraic system is obtained

$$\begin{aligned}&\varepsilon^2 h_{ijkpqr}\hat{\hat{U}}_i k_j k_k k_q k_r + \frac{\varepsilon}{2}(h_{ijpqr} - h_{pqijr}) I \hat{\hat{U}}_i k_j k_q k_r + h_{ijpq}\hat{\hat{U}}_i k_j k_q \\ &\quad = \hat{\hat{f}}_p + \rho_M \omega^2 \hat{\hat{U}}_p \qquad + \varepsilon \rho_M I \omega^2 I_{ipj} \hat{\hat{U}}_i k_j + \varepsilon^2 \rho_M \omega^2 I_{iqpj} \hat{\hat{U}}_i k_j k_q \\ &\qquad - \varepsilon^3 \rho_M I \omega^2 I_{iqpjs} \hat{\hat{U}}_i k_j k_q k_s + \varepsilon^4 \rho_M \omega^2 I_{iqjpsr} \hat{\hat{U}}_i k_j k_q k_s k_r. \end{aligned} \tag{25}$$

In case of vanishing body forces (wave propagation) the resulting homogeneous problem takes the form of an eigenvalue problem

$$(h^k_{pi} - \omega^2 h^\omega_{pi})\hat{\hat{U}}_i = 0, \quad p = 1,2 \tag{26}$$

where ω is the eigenvalue and the constants h^k_{pi}, h^ω_{pi} are given as follows

$$\begin{aligned} h^k_{pi} &= \varepsilon^2 h_{ijkpqr} k^4 n_j n_k n_q n_r + \tfrac{\varepsilon}{2}(h_{ijpqr} - h_{pqijr}) I k^3 n_j n_q n_r + h_{ijpq} k^2 n_j n_q, \\ h^\omega_{pi} &= \rho_M (\delta_{ip} + I\varepsilon I_{ipj} k n_j + \varepsilon^2 I_{iqpj} k^2 n_j n_q - I\varepsilon^3 I_{iqpjs} k^3 n_j n_q n_s \\ &\quad + \varepsilon^4 I_{iqjpsr} k^4 n_j n_q n_s n_r), \end{aligned} \tag{27}$$

which provides the eigenvalue ω^2_ς and the eigenvector $\hat{\hat{\mathbf{U}}}^\varsigma$ ($\varsigma = 1,2$) as a solution. These eigenvectors define the polarization directions of plane waves propagating along $\mathbf{n}$ and differ from those obtained by (14) related to the equivalent classical continuum. In fact, the higher order elastic inertial constants in the considered non-local continuum induce dispersion and correction of polarization directions given by (14), in agreement with Boutin and Auriault (1993).

The solution of (25) is obtained as a linear combination of the eigenvectors $\hat{\hat{\mathbf{U}}}^\varsigma$, i.e. $\hat{\hat{U}}_p = A_\varsigma \hat{\hat{U}}^\varsigma_p$ where the constant A_ς is obtained by exploiting the orthogonality properties of the eigenvectors $\hat{\hat{\mathbf{U}}}^\varsigma$ and takes the form $A_\varsigma = \hat{\hat{f}}_p \hat{\hat{U}}^\varsigma_i / [(h^k_{pi} - \omega^2 h^\omega_{pi})\hat{\hat{U}}^\varsigma_p \hat{\hat{U}}^\varsigma_i]$ (ς not summed). Through an inverse Fourier transform is possible to determine the macro-displacement $U_p = \mathcal{F}^{-1}[\mathcal{F}^{-1}[\hat{\hat{U}}_p]] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{+\infty}\hat{\hat{U}}_p e^{I k \mathbf{n}\cdot\mathbf{x}} k\,dk\,d\vartheta) e^{I\omega t} d\omega$ with $n_1 = \cos\vartheta$ and $n_2 = \sin\vartheta$.

It should be noted that the variational-asymptotic method here considered leads to approximate solutions of the real solution for a given parameter ε. Although this parameter has been assumed to be small, as detailed in the previous session, it would seem that, on the basis of the final discussion in Smyshlyaev and Cherednichenko (2000), a good approximation can be obtained even for small but not too small values of the parameter epsilon (see Appendix E).

Considering the rather common case of centro-symmetric periodic cell orthotropic material phases, the elastic wave propagation along the orthotropy direction $\mathbf{e}_\beta$ ($\beta = 1,2$) is described by the non-zero components of the macro-displacement vector $U_\alpha(x_\beta, t)$ ($\alpha = 1,2$) (Fig. 2) which are solution of the PDE

$$C_{\beta\alpha\beta\alpha} U_{\alpha,\beta\beta} - S_{\alpha\beta\beta\alpha\beta\beta} U_{\alpha,\beta\beta\beta\beta} = \rho_M \ddot{U}_\alpha - J_{\alpha\beta\alpha\beta}\ddot{U}_{\alpha,\beta\beta} + J_{\alpha\beta\beta\alpha\beta\beta}\ddot{U}_{\alpha,\beta\beta\beta\beta}, \tag{28}$$

indices not summed. Denoting $\hat{c}^\alpha_\beta = \sqrt{C_{\beta\alpha\beta\alpha}/\rho_M}$ the velocity of the compressional ($\alpha = \beta$) and shear ($\alpha \neq \beta$) waves along direction $\mathbf{e}_\beta$ in the classical equivalent continuum ($S_{\alpha\beta\beta\alpha\beta\beta} = J_{\alpha\beta\alpha\beta} = 0$ in Eq. (28)), $\lambda^\alpha_\beta = \sqrt{S_{\alpha\beta\beta\alpha\beta\beta}/C_{\beta\alpha\beta\alpha}}$ the extensional ($\alpha = \beta$) and shearing ($\alpha \neq \beta$) characteristic lengths of the heterogeneous material, $J_{\alpha\beta\alpha\beta} = I_{\alpha\beta\alpha\beta}\rho_M \varepsilon^2$ the second-order inertia tensor ($I_{\alpha\beta\alpha\beta}$ being fourth-order tensor depending on the geometrical and mechanical properties of the cell) and $J_{\alpha\beta\beta\alpha\beta\beta} = I_{\alpha\beta\beta\alpha\beta\beta}\rho_M\varepsilon^4$ the fourth-order inertia tensor ($I_{\alpha\beta\beta\alpha\beta\beta}$ being sixth-order tensor depending on the geometrical and mechanical properties of the cell), the displacement equation of motion may be written in the form

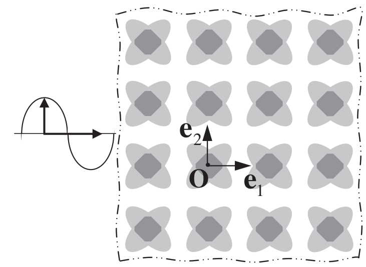

Fig. 2. Shear wave propagation along the orthotropy direction $\mathbf{e}_1$.

$$-(\lambda^\alpha_\beta \hat{c}^\alpha_\beta)^2 U_{\alpha,\beta\beta\beta\beta} + \hat{c}^{\alpha 2}_\beta U_{\alpha,\beta\beta} = \ddot{U}_\alpha - I_{\alpha\beta\alpha\beta}\varepsilon^2 \ddot{U}_{\alpha,\beta\beta} + I_{\alpha\beta\beta\alpha\beta\beta}\varepsilon^4 \ddot{U}_{\alpha,\beta\beta\beta\beta}. \tag{29}$$

It is worth to note that the eigenvalues of (14), related the wave propagation along the orthotropy axis $\mathbf{e}_\beta$ are identical to the wave velocity $\hat{c}^\alpha_\beta = \sqrt{C_{\beta\alpha\beta\alpha}/\rho_M}$.

To obtain the dispersion functions, let us seek solutions to Eq. (29) of the form $U_\alpha(x_\beta, t) = A\exp[I(kx_\beta - \omega t)]$, where k is the wave-number and ω is the angular frequency. The wavelength and the phase velocity of the in-plane waves along direction $\mathbf{e}_\beta$ are $\lambda = 2\pi/k$ and $c^\alpha_\beta = \omega/k$, respectively. The dispersion function corresponding to both the longitudinal ($\alpha = \beta$) and the transverse ($\alpha \neq \beta$) oscillatory motion of the derived equivalent continuum take the following form

$$\begin{aligned}\omega &= k\hat{c}^\alpha_\beta \sqrt{\frac{1 + (k\lambda^\alpha_\beta)^2}{1 + I_{\alpha\beta\alpha\beta}(k\varepsilon)^2 + I_{\alpha\beta\beta\alpha\beta\beta}(k\varepsilon)^4}} \\ &= k\hat{c}^\alpha_\beta \sqrt{\frac{1 + 4\pi^2(\lambda^\alpha_\beta/\lambda)^2}{1 + 4\pi^2 I_{\alpha\beta\alpha\beta}(\varepsilon/\lambda)^2 + 16\pi^4 I_{\alpha\beta\beta\alpha\beta\beta}(\varepsilon/\lambda)^4}}.\end{aligned} \tag{30}$$

From Eq. (30) it results that for large wavelengths ($\lambda \to \infty$) the angular frequency tends to the value related to the classical continuum, i.e. $\omega \to k\hat{c}^\alpha_\beta$. If the third and fourth order inertial terms of are neglected in (19), the dispersion relation (30) takes the form obtained by Bacigalupo and Gambarotta (2012a).

5. Illustrative example: Dispersion relations for layered materials

To evaluate the reliability of the proposed approach and its validity limits, a layered bi-material with orthotropic phases having the orthotropy axes parallel to the layering direction $\mathbf{e}_1$ is considered. In this case is possible to obtain analytical solutions for the fluctuation functions and to compare the dispersion functions with the exact ones by Rytov (1956) and Sun et al. (1968) (based on the Floquet–Bloch theorem). A further approximate solution is considered by the simplified model obtained for $I_{\alpha\beta\beta\alpha\beta\beta} = 0$ (by a third order truncation of the equation of motion (25) in analogy to Bacigalupo and Gambarotta (2012a). Finally, previous solutions proposed by Bacigalupo and Gambarotta (2012a,b) are compared. The inertia tensors components related to the waves propagation along the orthotropy directions are given in Appendix A.

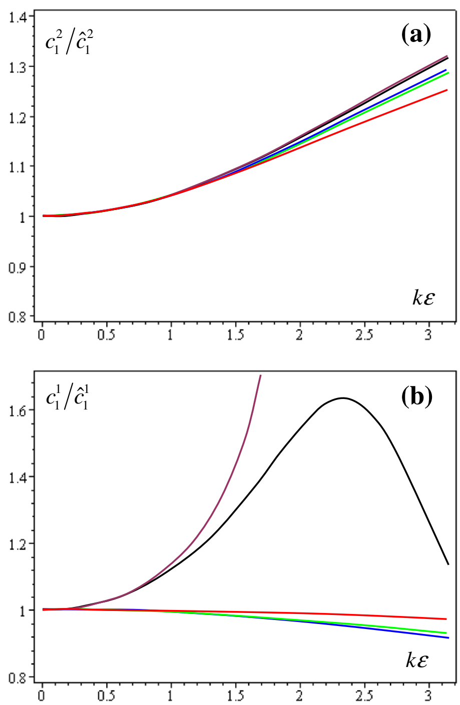

Fig. 3. Shear (a) and compressional (b) waves along the layering direction $\mathbf{e}_1$: non-dimensional phase velocities versus non-dimensional wave-number. Red line: heterogeneous material (Sun et al., 1968); Green line: proposed model; Blue line: simplified proposed model; Violet line: Bacigalupo and Gambarotta (2012a); Black line: Bacigalupo and Gambarotta (2012b). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

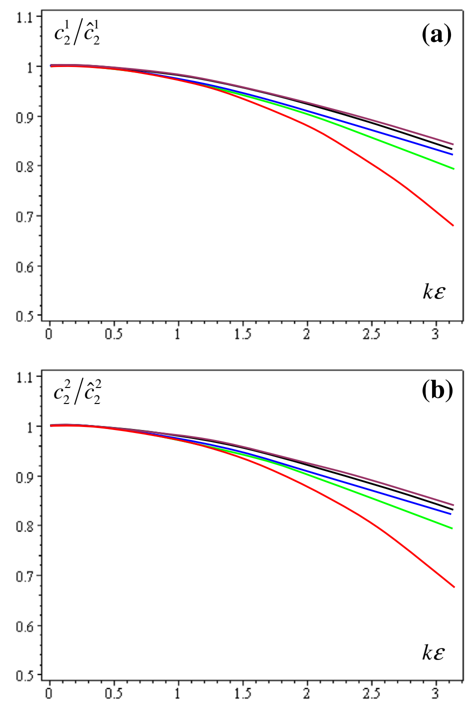

Fig. 4. Shear (a) and compressional (b) waves along direction $\mathbf{e}_2$ normal to the layers: non-dimensional phase velocities versus non-dimensional wave-number. Red line: heterogeneous material (Rytov, 1956); Green line: proposed model; Blue line: simplified proposed model; Violet line: Bacigalupo and Gambarotta (2012a); Black line: Bacigalupo and Gambarotta (2012b). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The two-phase stratified composite consists of perfectly bonded isotropic elastic layers arranged periodically having equal thickness $\varepsilon/2 = 0.5$ mm (i.e. $\eta = 1$), that was already analysed by Bacigalupo and Gambarotta (2010). The layered composite is assumed in plane strain condition, the Young moduli and the Poisson ratio of the layers are $E_a = 210000$ MPa, $E_b = 21000$ MPa, $\nu_a = \nu_b = 0.3$, respectively. The mass density for the components are $\rho_a = 8000$ kg/m^3, $\rho_b = 800$ kg/m^3. The unit cell is considered having characteristic size $\varepsilon = 1$ mm. The characteristics lengths, which depend by the dynamic corrector terms through the overall elasticity moduli, are: $\lambda_{Sh-1} = \lambda_1^2 = 0.353$ mm, $\lambda_{Sh-2} = \lambda_2^1 = 0.118$ mm, $\lambda_{Com-1} = \lambda_1^1 = 0.121$ mm, $\lambda_{Com-2} = \lambda_2^2 = 0.118$ mm. The velocity of the shear and compressional waves along direction $\mathbf{e}_\beta$ in the classical equivalent continuum are: $\hat{c}_1^2 = \hat{c}_2^1 = 1827$m/s, $\hat{c}_1^1 = 5567$m/s, $\hat{c}_2^2 = 3418$m/s. The overall mass density is $\rho_M = 4400$ kg/m^3 and the non-vanishing components of the micro-inertia tensors required to represent the compressional and shear waves along the orthotropy axes are: $I_{2121} = 0.03387$, $I_{1212} = 0.06973$, $I_{1111} = 0.03245$, $I_{2222} = 0.06973$, $I_{211211} = 1.652 \times 10^{-4}$, $I_{122122} = 1.278 \times 10^{-3}$, $I_{111111} = 3.627 \times 10^{-4}$, $I_{222222} = 1.278 \times 10^{-3}$.

The dimensionless phase velocity $c_\beta^\alpha/\hat{c}_\beta^\alpha$ of both shear and compressional waves are shown in the diagrams of Fig. 3 and Fig. 4 in terms of the dimensionless wave-number $k\varepsilon \in [0, \pi]$. In this interval, which is commonly considered in the literature (see Andrianov et al., 2008, 2011; Wang and Sun, 2002), are included values $k\varepsilon$ that do not seem to satisfy the smallness condition $\varepsilon \ll \lambda$, i.e. $k\varepsilon \ll 2\pi$. However, the reliability of the approximate technique here proposed may be assessed by comparing the dispersion curves by the proposed approach with those by the exact theory of Floquet–Bloch. The validity of the approximate model for given values of non-dimensional wave-number $k\varepsilon$ may be established on the basis of a sufficiently small difference (e.g. less than 5%) between the values of the dimensionless velocity by the proposed model with those obtained from the exact theory of Floquet–Bloch (see Rytov, 1956; Sun et al., 1968).

The dispersion functions related to shear and compressional waves (Rytov, 1956; Sun et al., 1968) are shown by the red lines in Figs. 3 and 4. The corresponding functions given by the proposed dynamic homogenization procedure are drawn in green, while the blue lines represent the results from the simplified proposed model obtained by truncating at the third order in ε the equation of motion Finally, the violet and the black lines represent the results from previous approaches proposed by Bacigalupo and Gambarotta (2012a,b). From the diagrams of Fig. 3 the dispersion functions obtained by the proposed model and by the simplified proposed model are in good agreement with those of the heterogeneous material obtained by the Floquet–Bloch theory. The dispersion curves for dispersive waves traveling along the layering direction are shown for a wider range of non-dimensional wave-number in Appendix E in order to assess the error by the dynamic homogenization model here proposed with respect to the exact solution by the Floquet–Bloch theory. Moreover, from the diagrams of Fig. 4 the dispersion functions by the proposed models for waves travelling along $\mathbf{e}_2$ are in good agreement with those ones by the Floquet–Bloch theory.

If one considers the direction of propagation $\mathbf{e}_2$, because of the periodicity of the material along that direction, a better approximation may be obtained for the dispersion functions in the range $k\varepsilon > \pi$. This approximation is derived as solution of Eq. (29) in the form $U_\alpha(x_\beta, t) = A\exp[I((k - 2\pi n/\varepsilon)x_\beta - \omega t)]$ (with $n \in \mathbb{Z}$) that leads to the dispersion function

$$\omega = k\hat{c}_\beta^\alpha \left|1 - \frac{2\pi n}{k\varepsilon}\right| \sqrt{\frac{1 + (k\varepsilon - 2\pi n)^2 (\lambda_\beta^\alpha/\varepsilon)^2}{1 + I_{\alpha\beta\alpha\beta}(k\varepsilon - 2\pi n)^2 + I_{\alpha\beta\beta\alpha\beta\beta}(k\varepsilon - 2\pi n)^4}}. \tag{31}$$

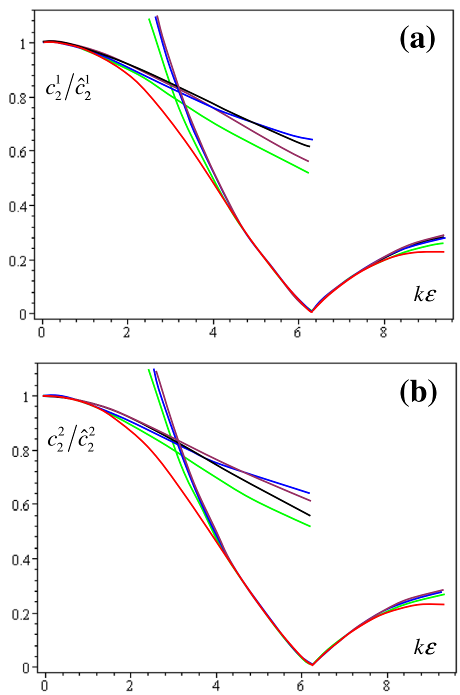

Fig. 5. Shear (a) and compressional (b) waves travelling along direction $\mathbf{e}_2$: non-dimensional phase velocities for $n = 0$ and $n = 1$ (see dispersion function (37)) versus non-dimensional wave-number.

This function is in good agreement with those obtained through the Floquet–Bloch theorem in Rytov, 1956, for $\pi(2n-1) < k\varepsilon < \pi(2n+1), \forall n \in \mathbb{Z}$. Namely, the exact dispersion curves are approximated around the points $k\varepsilon = 2n\pi$ by two branches, each defined within an interval of length π. Therefore, the dispersion functions associated to $n = 0$ and $n = 1$ are shown in the diagrams of Fig. 5. In this case the results obtained for high values of $k\varepsilon$ are justified by the translation of the phase of the wave that leads to the dispersion function (31).

As a second example, the two-phase stratified composite consisting of isotropic elastic layers arranged periodically and having equal thickness $\varepsilon/2 = 0.5$ mm (i.e. $\eta = 1$) is considered. The layered composite is assumed in plane strain condition, the Young moduli and the Poisson ratio of the layers are $E_a = 210000$ MPa, $\nu_a = 0.3$, $E_b = 1000$ MPa, $\nu_b = 0.49$, respectively. For both phases, the same mass density is assumed equal to $\rho = 2000\ \mathrm{kg/m^3}$. The unit cell is considered having characteristic size $\varepsilon = 1$ mm. The characteristics lengths are independent on the dynamic corrector and are: $\lambda_{Sh-1} = \lambda_1^2 = 1.899$ mm, $\lambda_{Sh-2} = \lambda_2^1 = 0$, $\lambda_{Com-1} = \lambda_1^1 = 0.051$ mm, $\lambda_{Com-2} = \lambda_2^2 = 0$. The velocity of the shear and compressional waves along direction $\mathbf{e}_\beta$ in the classical equivalent continuum are: $\hat{c}_1^2 = \hat{c}_2^1 = 578$m/s, $\hat{c}_1^1 = 8112$m/s, $\hat{c}_2^2 = 4017$m/s. The overall mass density is $\rho_M = 2000\ \mathrm{kg/m^3}$ and the non-vanishing components of the micro-inertia tensors required to represent the compressional and shear waves along the orthotropy axes are: $I_{1212} = I_{2121} = 0.02049$, $I_{1111} = 0.00254$, $I_{2222} = 0.01635$, $I_{211211} = 2.532 \times 10^{-4}$, $I_{122122} = 5.122 \times 10^{-4}$, $I_{111111} = 15.993$, $I_{222222} = 4.087 \times 10^{-4}$. Since the mass density is homogeneous throughout the layered material, it follows that the dynamic terms of the micro-fluctuation functions $N_{ipqr}^{2-D}(\xi)$ are vanishing. In this case, both the overall elastic moduli (see Eq. (23)) and the inertia terms (see Eq. (24)) are independent on the mass density. It is worth to note that the inertial coefficient I_{111111} associated with the propagation of the compressional waves along the layering direction $\mathbf{e}_1$ takes high values in comparison to the other terms of the same order, so explaining the different behavior of the green curve in Fig. 6.b.

The dimensionless phase velocity $c_\beta^\alpha/\hat{c}_\beta^\alpha$ of both shear and compressional waves are shown in the diagrams of Fig. 6 in terms of the dimensionless wave-number $k\varepsilon$. In these diagrams, the exact dispersion functions for the layered material (Rytov, 1956;

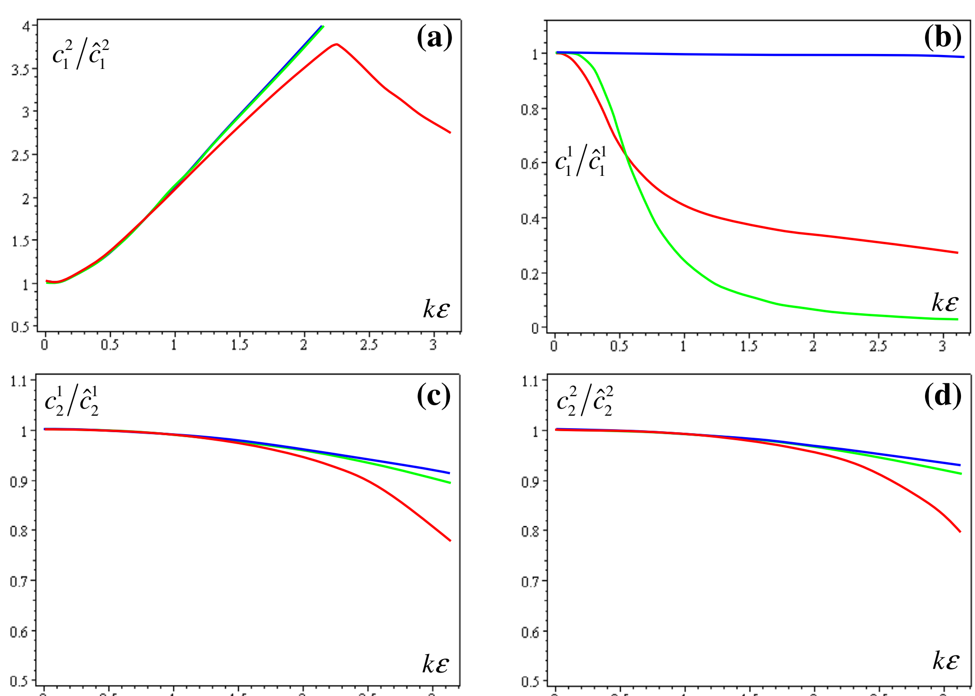

Fig. 6. Shear and compressional waves: non-dimensional phase velocities versus non-dimensional wave-number. Red line: layered material (Sun et al., 1968; Rytov, 1956); Green line: homogenized proposed model; Blue line: simplified proposed model. (a) S. waves in $\mathbf{e}_1$ direction; (b) C. waves in $\mathbf{e}_1$ direction; (c) S. waves in $\mathbf{e}_2$ direction; (d) C. waves in e_2 direction. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

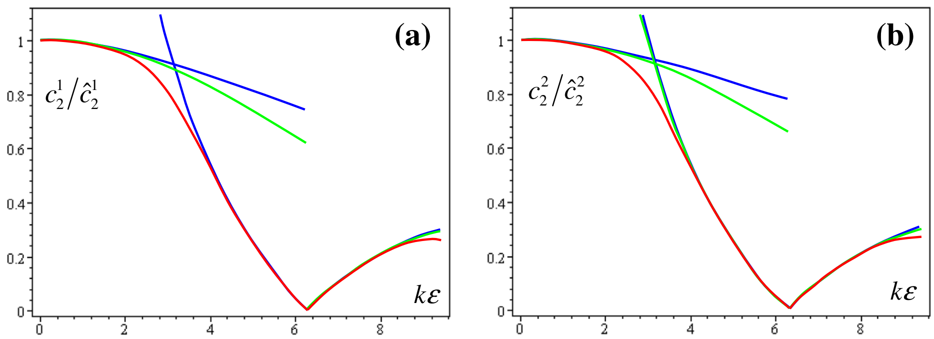

Fig. 7. Shear (a) and compressional (b) waves along direction $\mathbf{e}_2$: non-dimensional phase velocities for $n = 0$ and $n = 1$ (see dispersion function (37)) versus non-dimensional wave-number.

Sun et al., 1968) are represented by the red lines, while those obtained by the present model (which coincides with Bacigalupo and Gambarotta (2012b)) are represented by green lines. Finally, the functions obtained by the simplified method (which coincides with Bacigalupo and Gambarotta, 2012a) are represented by the blue lines. From the diagrams of Fig. 6 it emerges that the difference between the velocity of dispersive waves along the direction of layers and the corresponding one in the classical continuum is remarkable. Conversely, the phase velocity of the dispersive waves along the direction normal to the layers differs very little from the velocity obtained by considering the classical continuum. Furthermore, it can be seen that the inertial constant $I_{\alpha\beta\beta\alpha\beta\beta}$ has a marked influence in the case of compressional waves along the layers ($\alpha = \beta = 1$) (see Fig. 6b). In fact, if this term is considered the dispersion curve of the heterogeneous material is well approximated for values less than $k\varepsilon \approx 0.5$ and with an error less than 10% for $k\varepsilon \approx 0.7$.

Finally, the dispersion functions related to wave propagation along direction $\mathbf{e}_2$ given by Eq. (31) for $n = 0$ and $n = 1$ are shown in the diagrams of Fig. 7. Here, the results provided by the proposed model appears to be in good agreement with those obtained through the Floquet–Bloch theory by Rytov (1956), for $\pi(2n-1) < k\varepsilon < \pi(2n+1) \forall n \in \mathbb{Z}$.

6. Conclusions

A dynamic homogenization procedure for the analysis of wave propagation in materials with periodic microstructure has been proposed to obtain a sufficiently accurate simulation of the lowest (acoustic) branch of the Bloch spectrum for a wide enough range of wavelengths. By a reformulation of the variational-asymptotic homogenization technique recently proposed by Bacigalupo and Gambarotta (2012a), a second gradient continuum model has been derived, which provides a good description of the dispersion curves. The improvement of the accuracy of the method is obtained through a more detailed description of the multi-scale kinematics by means of appropriate micro-fluctuation functions of the displacement field depending on the material microstructure. These functions are derived by taking the displacements as an asymptotic expansion and so obtaining by the equation of motion a recurrent sequence of cell BVPs. The solutions of these problems are obtained as the superposition of a static contribution (similar to that determined in Bacigalupo and Gambarotta (2012a,b)), that depends on the elastic properties of the phases, and of a dynamic contribution that also depends on the mass density of the phases. Notably, the dynamic contributions are proportional to the even powers of the phase velocity and consequently the micro-fluctuation functions also depend on the direction of propagation, i.e. the angular frequency and the wave-vector. Moreover, in the particular case of homogeneous mass density the dynamic contributions of micro-fluctuation functions vanish. The macro-displacement field that satisfies the average equations of infinite order in the transformed space provides, through the proposed downscaling, a micro-displacement field that fulfils the equation of motion at the fine scale. On the other side, the micro-displacement field downscaled from the second-gradient continuum derived by the variational-asymptotic approach only fulfils the weak form of the equation of motion. The equation of motion at the coarse scale is obtained through the Hamilton's principle in terms of the overall elastic moduli and of inertial terms. The higher order terms depend on the static and dynamic contributions to the micro-fluctuation functions. With respect to previous formulations (see Bacigalupo and Gambarotta (2012a,b), among the others), both the higher order elastic moduli and the inertial terms also depend on dynamic correctors.

To evaluate the reliabily of the proposed approach and its validity limits, a bimaterial periodically-layered composite with orthotropic phases having a orthotropy axis parallel to the layering direction has been considered. For this composite material, the micro-fluctuation functions and the second order overall constants have been analytically determined. The obtained dispersion functions for shear and compressional waves along and normal to the layering direction are in good agreement with those from the Floquet–Bloch approach. Notably, it is noted that for waves travelling along the layering direction, also the dispersion functions obtained from the simplified model (in which some higher order inertial terms are ignored) are in good agreement with those obtained by the Floquet–Bloch theory. The proposed method appears to be more reliable in the case where the difference in mass density between the phases is higher, with a good approximation to the exact solution for dimensionless wave-number $k\varepsilon \leqslant 3\pi$. In the case of homogeneous mass density, good approximations are obtained for shear waves if $k\varepsilon \leqslant 2\pi$ and for compressional waves if $k\varepsilon \leqslant \pi$, respectively. For waves traveling along the normal to the layers, the first branch of the dispersion functions obtained by the second-gradient model well approximates the one by Floquet–Bloch solution for every value of the dimensionless wave-number $k\varepsilon$. In fact, in this case it is possible to determine a sequence of dispersion functions, shifted by $k\varepsilon = 2\pi$, whose envelope well approximates the first branch of the dispersion functions by the Floquet–Bloch theory.

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Appendix A. Higher order inertia tensors for orthotropic layered materials

Let us consider a layered body obtained as a unbounded d_2-periodic arrangement of two different layers having thickness a and b (here $d_2 = \varepsilon = a + b$ and $\eta = a/b$ are defined), respectively. The phases are assumed homogeneous and orthotropic, with an orthotropic axis coincident with the layering direction $\mathbf{e}_1$. The micro-fluctuation functions $\mathbf{N}^1(\xi)$ and $\mathbf{N}^2(\xi)$ (with components $N^1_{ikl}(\xi)$ and $N^2_{iklp}(\xi)$) are analytically obtained through the solution of the *cell problems* formulated in Section 3. These functions, because the particular properties of symmetry of the microstructure are dependent only on the fast variable ξ_2. The non-vanishing micro-fluctuation functions $N^1_{ikl}(\xi)$ obtained by solving the first cell problem are:

$$
\begin{aligned}
&N^{1\_a}_{211} = \frac{C^b_{1122}-C^a_{1122}}{C^a_{2222}+\eta C^b_{2222}}\xi^a_2, \quad N^{1\_b}_{211} = \eta\frac{C^a_{1122}-C^b_{1122}}{C^a_{2222}+\eta C^b_{2222}}\xi^b_2,\\
&N^{1\_a}_{222} = \frac{C^b_{2222}-C^a_{2222}}{C^a_{2222}+\eta C^b_{2222}}\xi^a_2, \quad N^{1\_b}_{222} = \eta\frac{C^a_{2222}-C^b_{2222}}{C^a_{2222}+\eta C^b_{2222}}\xi^b_2,\\
&N^{1\_a}_{112} = N^{1\_a}_{121} = \frac{C^b_{1212}-C^a_{1212}}{C^a_{1212}+\eta C^b_{1212}}\xi^a_2, \quad N^{1\_b}_{112} = N^{1\_b}_{121} = \eta\frac{C^a_{1212}-C^b_{1212}}{C^a_{1212}+\eta C^b_{1212}}\xi^b_2,
\end{aligned}
\tag{32}
$$

where $\xi^a_2 \in \left[-\frac{\eta}{2(\eta+1)}, \frac{\eta}{2(\eta+1)}\right]$ and $\xi^b_2 \in \left[-\frac{1}{2(\eta+1)}, \frac{1}{2(\eta+1)}\right]$ are two non-dimensional vertical coordinates centered in each layers. The non-vanishing functions $N^{2\_S}_{ipqr}(\xi)$ and $N^{2\_D}_{ipqr}(\xi)$ (terms of the micro-fluctuation functions $N^2_{ipqr} = N^{2\_S}_{ipqr} + c^2 N^{2\_D}_{ipqr}$) related to the elastic wave propagation along orthotropy directions are obtained by solving the second cell problem and take the form

$$
\begin{aligned}
&N^{2\_Sa}_{2211} = A^{2S}_{2211}\xi^{a2}_2 + A^{1S}_{2211}, \quad N^{2\_Sb}_{2211} = -\eta\frac{C^a_{2222}C^b_{1122}}{C^b_{2222}C^a_{1122}}A^{2S}_{2211}\xi^{b2}_2 + B^{1S}_{2211},\\
&N^{2\_Sa}_{2222} = A^{2S}_{2222}\xi^{a2}_2 + A^{1S}_{2222}, \quad N^{2\_Sb}_{2222} = -\eta A^{2S}_{2222}\xi^{b2}_2 - \frac{2\eta+1}{\eta+2}A^{1S}_{2222},\\
&N^{2\_Sa}_{1122} = A^{2S}_{1122}\xi^{a2}_2 + A^{1S}_{1122}, \quad N^{2\_Sb}_{1122} = -\eta A^{2S}_{1122}\xi^{b2}_2 - \frac{2\eta+1}{\eta+2}A^{1S}_{1122},\\
&N^{2\_Sa}_{1111} = A^{2S}_{1111}\xi^{a2}_2 + A^{1S}_{1111}, \quad N^{2\_Sb}_{1111} = B^{2S}_{1111}\xi^{b2}_2 + B^{1S}_{1111},\\
&N^{2\_Da}_{2211} = (\rho_a - \rho_b)(A^{2D}_{2211}\xi^{a2}_2 + A^{1D}_{2211}),\\
&N^{2\_Db}_{2211} = (\rho_a - \rho_b)\left(-\eta\frac{C^a_{2222}}{C^b_{2222}}A^{2D}_{2211}\xi^{a2}_2 + B^{1D}_{2211}\right),\\
&N^{2\_Da}_{1122} = (\rho_a - \rho_b)(A^{2D}_{1122}\xi^{a2}_2 + A^{1D}_{1122}),\\
&N^{2\_Db}_{1122} = (\rho_a - \rho_b)\left(-\eta\frac{C^a_{1212}}{C^b_{1212}}A^{2D}_{1122}\xi^{a2}_2 + B^{1D}_{1122}\right),\\
&N^{2\_Da}_{2222} = N^{2\_Da}_{2211}, \quad N^{2\_Db}_{2222} = N^{2\_Db}_{2211}, \quad N^{2\_Da}_{1111} = N^{2\_Da}_{1122}, \quad N^{2\_Db}_{1111} = N^{2\_Db}_{1122},
\end{aligned}
\tag{33}
$$

where the constants A^{2S}_{2211}, A^{1S}_{2211}, B^{1S}_{2211}, A^{2S}_{2222}, A^{1S}_{2222}, A^{2S}_{1122}, A^{1S}_{1122}, A^{2S}_{1111}, A^{1S}_{1111}, B^{2S}_{1111}, B^{1S}_{1111}, A^{2D}_{2211}, A^{1D}_{2211}, B^{1D}_{2211}, A^{2D}_{1122}, A^{1D}_{1122}, B^{1D}_{1122} are given in Appendix B.

In case of elastic wave propagation along the orthotropy direction $\mathbf{e}_\beta$ ($\beta = 1, 2$), the overall inertial constants $I_{\alpha\beta\alpha\beta}$ and $I_{\alpha\beta\beta\alpha\beta\beta}$ in the dispersion relation (30) are specialized, assuming homogeneous mass density of the constituents, and take the form:

- Shear waves along $\mathbf{e}_1$

$$
\begin{aligned}
I_{2121} &= I^S_{2121} + c^2_\varsigma I^D_{2121} = \left\langle \rho(N^1_{121}N^1_{121}) - 2N^{2\_S}_{2211}\right\rangle/\rho_M - c^2_\varsigma\left\langle 2\rho N^{2\_D}_{2211}\right\rangle/\rho_M,\\
I_{211211} &= I^S_{211211} + c^2_\varsigma I^{D1}_{211211} + c^4_\varsigma I^{D2}_{211211} = \left\langle \rho N^{2\_S}_{2211}N^{2\_S}_{2211}\right\rangle/\rho_M\\
&\quad + c^2_\varsigma\left\langle 2\rho N^{2\_S}_{2211}N^{2\_D}_{2211}\right\rangle/\rho_M + c^4_\varsigma\left\langle \rho N^{2\_D}_{2211}N^{2\_D}_{2211}\right\rangle/\rho_M,
\end{aligned}
\tag{34}
$$

in which

$$
\begin{aligned}
I^S_{2121} &= \frac{\eta^2(C^a_{1212}-C^b_{1212})(\rho_a I^{S\_a}_{2121}+\rho_b I^{S\_b}_{2121})}{12(\eta+1)^3 C^a_{2222}C^b_{2222}(C^a_{1212}+\eta C^b_{1212})^2(\eta\rho_a+\rho_b)},\\
I^D_{2121} &= \frac{\eta^2(\rho_b-\rho_a)^2(\eta C^b_{2222}+C^a_{2222})}{6(\eta+1)^4 C^a_{2222}C^b_{2222}(\eta\rho_a+\rho_b)},\\
I^S_{211211} &= \frac{\eta^2(C^a_{1212}-C^b_{1212})^2(\rho_a I^{S\_a}_{211211}+\rho_b I^{S\_b}_{211211})}{720(\eta+1)^6(C^a_{2222}C^b_{2222})^2(C^a_{1212}+\eta C^b_{1212})^2(\eta\rho_a+\rho_b)},\\
I^{D1}_{211211} &= -\frac{\eta^2(\rho_b-\rho_a)(C^a_{1212}-C^b_{1212})(\rho_a I^{D1\_a}_{211211}+\rho_b I^{D1\_b}_{211211})}{360(\eta+1)^7(C^a_{2222}C^b_{2222})^2(C^a_{1212}+\eta C^b_{1212})^2(\eta\rho_a+\rho_b)},\\
I^{D2}_{211211} &= \frac{\eta^2(\rho_b-\rho_a)^2(\rho_a I^{D2\_a}_{211211}+\rho_b I^{D2\_b}_{211211})}{720(\eta+1)^8(C^a_{2222}C^b_{2222})^2(\eta\rho_a+\rho_b)},
\end{aligned}
$$

and the constants $I^{S\_a}_{2121}$, $I^{S\_b}_{2121}$, $I^{S\_a}_{211211}$, $I^{S\_b}_{211211}$, $I^{D1\_a}_{211211}$, $I^{D1\_b}_{211211}$, $I^{D2\_a}_{211211}$, $I^{D2\_b}_{211211}$ are given in Appendix C.

- Compressional waves along $\mathbf{e}_2$

$$
\begin{aligned}
I_{2222} &= I^S_{2222} + c^2_\varsigma I^D_{2222}\\
&= \left\langle \rho(N^1_{222}N^1_{222}) - 2N^{2\_S}_{2222}\right\rangle\Big/\rho_M - c^2_\varsigma\left\langle 2\rho N^{2\_D}_{2222}\right\rangle\Big/\rho_M,\\
I_{222222} &= I^S_{222222} + c^2_\varsigma I^{D1}_{222222} + c^4_\varsigma I^{D2}_{222222}\\
&= \left\langle \rho N^{2\_S}_{2222}N^{2\_S}_{2222}\right\rangle\Big/\rho_M + c^2_\varsigma\left\langle 2\rho N^{2\_S}_{2222}N^{2\_D}_{2222}\right\rangle\Big/\rho_M\\
&\quad + c^4_\varsigma\left\langle \rho N^{2\_D}_{2222}N^{2\_D}_{2222}\right\rangle\Big/\rho_M,
\end{aligned}
\tag{35}
$$

in which

$$
I^S_{2222} = \frac{\eta^2(C^b_{2222}-C^a_{2222})(2\rho_b\eta C^b_{2222}-\rho_a\eta C^b_{2222}+\rho_b C^b_{2222}-2\rho_a C^a_{2222}-\rho_a\eta C^a_{2222}+\rho_b C^a_{2222})}{12(\eta+1)^2(C^a_{2222}+\eta C^b_{2222})^2(\eta\rho_a+\rho_b)},
$$

$$
I^D_{2222} = I^D_{2121}, \quad I^{D2}_{222222} = I^{D2}_{211211}, \quad I^S_{222222} = \frac{\eta^2(C^a_{2222}-C^b_{2222})^2(\rho_a\eta^3+5\eta^2\rho_b+5\eta\rho_a+\rho_b)}{720(\eta+1)^4(C^a_{2222}+\eta C^b_{2222})^2(\eta\rho_a+\rho_b)},
$$

$$
I^{D1}_{222222} = -\frac{\eta^2(\rho_b-\rho_a)(C^a_{2222}-C^b_{2222})(\rho_a I^{D1\_a}_{222222}+\rho_b I^{D1\_b}_{222222})}{360(\eta+1)^6(C^a_{2222}C^b_{2222})^2(C^a_{2222}+\eta C^b_{2222})^2(\eta\rho_a+\rho_b)},
$$

where

$$
\begin{aligned}
I^{D1\_a}_{222222} &= \eta(\eta^2 C^b_{2222} + \eta^3 C^b_{2222} + 5\eta C^b_{2222} + 5C^b_{2222}),\\
I^{D1\_b}_{222222} &= 5\eta^3 C^b_{2222} + 5\eta^2 C^a_{2222} + \eta C^a_{2222} + C^a_{2222};.
\end{aligned}
$$

- Shear waves along $\mathbf{e}_2$

$$
\begin{aligned}
I_{1212} &= I^S_{1212} + c^2_\varsigma I^D_{1212} = \left\langle \rho(N^1_{112}N^1_{112}) - 2N^{2\_S}_{1122}\right\rangle/\rho_M - c^2_\varsigma\left\langle 2\rho N^{2\_D}_{1122}\right\rangle/\rho_M,\\
I_{122122} &= I^S_{122122} + c^2_\varsigma I^{D1}_{122122} + c^4_\varsigma I^{D2}_{122122} = \left\langle \rho N^{2\_S}_{1122}N^{2\_S}_{1122}\right\rangle/\rho_M\\
&\quad + c^2_\varsigma\left\langle 2\rho N^{2\_S}_{1122}N^{2\_D}_{1122}\right\rangle/\rho_M + c^4_\varsigma\left\langle \rho N^{2\_D}_{1122}N^{2\_D}_{1122}\right\rangle/\rho_M,
\end{aligned}
\tag{36}
$$

in which

$$
I^S_{1212} = \frac{\eta^2(C^b_{1212}-C^a_{1212})(2\rho_b\eta C^b_{1212}-\rho_a\eta C^b_{1212}+\rho_b C^b_{1212}-2\rho_a C^a_{1212}-\rho_a\eta C^a_{1212}+\rho_b C^a_{1212})}{12(\eta+1)^2(C^a_{1212}+\eta C^b_{1212})^2(\eta\rho_a+\rho_b)},
$$

$$
I^D_{1212} = \frac{\eta^2(\rho_a-\rho_b)^2(\eta C^b_{1212}+C^a_{1212})}{6(\eta+1)^4 C^a_{1212}C^b_{1212}(\eta\rho_a+\rho_b)}, \quad I^S_{122122} = \frac{\eta^2(C^b_{1212}-C^a_{1212})^2(\rho_a\eta^3+5\eta^2\rho_b+5\eta\rho_a+\rho_b)}{720(\eta+1)^4(C^a_{1212}+\eta C^b_{1212})^2(\eta\rho_a+\rho_b)},
$$

$$
\begin{aligned}
I^{D1}_{122122} &= \frac{\eta^2(\rho_b-\rho_a)(C^b_{1212}-C^a_{1212})(\rho_a I^{D1\_a}_{122122}+\rho_b I^{D1\_b}_{122122})}{360(\eta+1)^6 C^a_{1212}C^b_{1212}(C^a_{1212}+\eta C^b_{1212})(\eta\rho_a+\rho_b)},\\
I^{D2}_{122122} &= \frac{\eta^2(\rho_b-\rho_a)^2(\rho_a I^{D2\_a}_{122122}+\rho_b I^{D2\_b}_{122122})}{720(\eta+1)^8(C^a_{1212}C^b_{1212})^2(\eta\rho_a+\rho_b)},
\end{aligned}
$$

where

$$
\begin{aligned}
I^{D1\_a}_{122122} &= \eta(5\eta C^b_{1212} + \eta^3 C^b_{1212} + \eta^2 C^b_{1212} + 5C^a_{1212}),\\
I^{D1\_b}_{122122} &= 5\eta^3 C^b_{1212} + 5\eta^2 C^a_{1212} + C^a_{1212} + \eta C^a_{1212},\\
I^{D2\_a}_{122122} &= \eta(2\eta^3 C^b_{1212}{}^2 + 10\eta C^b_{1212}C^a_{1212} + 5C^a_{1212}{}^2 + 6\eta^2 C^b_{1212}{}^2 + \eta^4 C^b_{1212}{}^2),\\
I^{D2\_b}_{122122} &= 10\eta^3 C^a_{1212}C^b_{1212} + C^a_{1212}{}^2 + 6\eta^2 C^a_{1212}{}^2 + 5\eta^4 C^b_{1212}{}^2 + 2\eta C^a_{1212}{}^2;
\end{aligned}
$$

Compressional waves along $\mathbf{e}_1$

$$
\begin{aligned}
I_{1111} &= I^{S}_{1111} + c_{\varsigma}^2 I^{D}_{1111} = \left\langle \rho(N^{1}_{211}N^{1}_{211}) - 2N^{2\_S}_{1111} \right\rangle/\rho_M \\
&\quad - c_{\varsigma}^2\left\langle 2\rho N^{2\_D}_{1111} \right\rangle/\rho_M, \\
I_{111111} &= I^{S}_{111111} + c_{\varsigma}^2 I^{D1}_{111111} + c_{\varsigma}^4 I^{D2}_{111111} \\
&= \left\langle \rho N^{2\_S}_{1111}N^{2\_S}_{1111} \right\rangle/\rho_M + c_{\varsigma}^2\left\langle 2\rho N^{2\_S}_{1111}N^{2\_D}_{1111} \right\rangle/\rho_M \\
&\quad + c_{\varsigma}^4\left\langle \rho N^{2\_D}_{1111}N^{2\_D}_{1111} \right\rangle/\rho_M,
\end{aligned}
\tag{37}
$$

in which

$$
\begin{aligned}
I^{S}_{1111} &= \frac{I^{S\_0}_{1111} + I^{S\_1}_{1111}\eta + I^{S\_2}_{1111}\eta^2 + I^{S\_3}_{1111}\eta^3}{12(\eta+1)^2(\eta\rho_a+\rho_b)}, I^{D}_{1111} = I^{D}_{1212}, \\
I^{S}_{111111} &= \frac{I^{S\_0}_{111111} + I^{S\_1}_{111111}\eta + I^{S\_2}_{111111}\eta^2 + I^{S\_3}_{111111}\eta^3 + I^{S\_4}_{111111}\eta^2 + I^{S\_5}_{111111}\eta^5}{240(\eta+1)^4(\eta\rho_a+\rho_b)}, \\
I^{D2}_{111111} &= I^{D2}_{122122}, \\
I^{D1}_{111111} &= \frac{(\rho_a-\rho_b)(I^{D1\_0}_{111111} + I^{D1\_1}_{111111}\eta + I^{D1\_2}_{111111}\eta^2 + I^{D1\_3}_{111111}\eta^3 + I^{D1\_4}_{111111}\eta^2 + I^{D1\_5}_{111111}\eta^5)}{120(\eta+1)^4(\eta\rho_a+\rho_b)},
\end{aligned}
$$

The constants $I^{S\_0}_{1111}$, $I^{S\_1}_{1111}$, $I^{S\_2}_{1111}$, $I^{S\_3}_{1111}$, $I^{S\_0}_{111111}$, $I^{S\_1}_{111111}$, $I^{S\_2}_{111111}$, $I^{S\_3}_{111111}$, $I^{S\_4}_{111111}$, $I^{S\_5}_{111111}$, $I^{D1\_0}_{111111}$, $I^{D1\_1}_{111111}$, $I^{D1\_2}_{111111}$, $I^{D1\_3}_{111111}$, $I^{D1\_4}_{111111}$, $I^{D1\_5}_{111111}$ are given in Appendix D.

For isotropic materials one obtains $C^{\alpha}_{1111} = C^{\alpha}_{2222} = \frac{\tilde{E}_\alpha}{1-\tilde{\nu}_\alpha^2}$, $C^{\alpha}_{1122} = \frac{\tilde{\nu}_\alpha\tilde{E}_\alpha}{1-\tilde{\nu}_\alpha^2}$, $C^{\alpha}_{1212} = \frac{\tilde{E}_\alpha}{2(1+\tilde{\nu}_\alpha)}$, (with $\alpha = a, b$), where $\tilde{E}_\alpha = \frac{E_\alpha}{1-\nu_\alpha^2}$, $\tilde{\nu}_\alpha = \frac{\nu_\alpha}{1-\nu_\alpha}$ in the case of plane-strain or $\tilde{E}_\alpha = E_\alpha$, $\tilde{\nu}_\alpha = \nu_\alpha$ in the case of plane-stress, E_α being the Young modulus and ν_α the Poisson ratio, respectively.

Appendix B

The constants A^{2S}_{2211}, A^{1S}_{2211}, B^{1S}_{2211}, A^{2S}_{2222}, A^{1S}_{2222}, A^{2S}_{1122}, A^{1S}_{1122}, A^{2S}_{1111}, A^{1S}_{1111}, B^{2S}_{1111}, B^{1S}_{1111}, A^{2D}_{2211}, A^{1D}_{2211}, B^{1D}_{2211}, A^{2D}_{1122}, A^{1D}_{1122}, B^{1D}_{1122} involved in (33) are written in the form

$$
\begin{aligned}
A^{2S}_{2211} &= \frac{C^a_{1122}(C^a_{1212}-C^b_{1212})}{2C^a_{2222}(C^a_{1212}+\eta C^b_{1212})}, \\
A^{1S}_{2211} &= \frac{\eta(C^a_{1122}C^b_{2222}\eta^2+3C^a_{1122}C^b_{2222}\eta+2C^b_{1122}C^a_{2222})(C^b_{1212}-C^a_{1212})}{24C^a_{2222}C^b_{2222}(\eta+1)^3(C^a_{1212}+\eta C^b_{1212})}, \\
B^{1S}_{2211} &= \frac{\eta(2C^a_{1122}C^b_{2222}\eta^2+3C^b_{1122}C^a_{2222}\eta+C^b_{1122}C^a_{2222})(C^a_{1212}-C^b_{1212})}{24C^a_{2222}C^b_{2222}(\eta+1)^3(C^a_{1212}+\eta C^b_{1212})}, \\
A^{2S}_{2222} &= \frac{C^a_{2222}-C^b_{2222}}{2(C^a_{2222}+\eta C^b_{2222})}, \quad A^{1S}_{2222} = \frac{\eta(\eta+2)(C^a_{2222}-C^b_{2222})}{24(\eta+1)^2(C^a_{2222}+\eta C^b_{2222})}, \\
A^{2S}_{1122} &= \frac{C^a_{1212}-C^b_{1212}}{2(C^a_{1212}+\eta C^b_{1212})}, \quad A^{1S}_{1122} = \frac{\eta(\eta+2)(C^a_{1212}-C^b_{1212})}{24(\eta+1)^2(C^a_{1212}+\eta C^b_{1212})}, \\
A^{1S}_{1111} &= \frac{\eta(A^{1S\_0}_{1111}+A^{1S\_1}_{1111}\eta+A^{1S\_2}_{1111}\eta^2+A^{1S\_3}_{1111}\eta^3)}{24C^a_{1212}C^b_{1212}(\eta+1)^4(C^a_{2222}+\eta C^b_{2222})}, \quad A^{2S}_{1111} = \frac{A^{2S\_0}_{1111}+A^{2S\_1}_{1111}\eta}{2C^a_{1212}(\eta+1)(C^a_{2222}+\eta C^b_{2222})}, \\
B^{1S}_{1111} &= \frac{\eta(-A^{1S\_0}_{1111}/2+B^{1S\_1}_{1111}\eta+B^{1S\_2}_{1111}\eta^2-2A^{1S\_3}_{1111}\eta^3)}{24C^a_{1212}C^b_{1212}(\eta+1)^4(C^a_{2222}+\eta C^b_{2222})}, \quad B^{2S}_{1111} = \frac{\eta(B^{2S\_0}_{1111}+B^{2S\_1}_{1111}\eta)}{2C^b_{1212}(\eta+1)(C^a_{2222}+\eta C^b_{2222})}, \\
A^{2D}_{2211} &= \frac{1}{2(\eta+1)C^a_{2222}}, \quad A^{1D}_{2211} = -\frac{\eta(C^b_{2222}\eta^2+3C^b_{2222}\eta+2C^a_{2222})}{24(\eta+1)^4C^a_{2222}C^b_{2222}}, \\
B^{2D}_{2211} &= -\frac{\eta}{2(\eta+1)C^b_{2222}}, \quad B^{1D}_{2211} = \frac{\eta(2C^b_{2222}\eta^2+3C^a_{2222}\eta+C^a_{2222})}{24(\eta+1)^4C^a_{2222}C^b_{2222}}, \\
A^{2D}_{1122} &= \frac{1}{2(\eta+1)C^a_{1212}}, \quad A^{1D}_{1122} = -\frac{\eta(C^b_{1212}\eta^2+3C^b_{1212}\eta+2C^a_{1212})}{24(\eta+1)^4C^a_{1212}C^b_{1212}}, \\
B^{2D}_{1122} &= -\frac{\eta}{2(\eta+1)C^b_{1212}}, \quad B^{1D}_{1122} = \frac{\eta(2C^b_{1212}\eta^2+3C^a_{1212}\eta+C^a_{1212})}{24(\eta+1)^4C^a_{1212}C^b_{1212}},
\end{aligned}
$$

where

$$
\begin{aligned}
A^{1S\_0}_{1111} &= 2C^a_{1212}(C^a_{1111}C^a_{2222} - C^b_{1111}C^a_{2222} + C^b_{1212}C^b_{1122} - {C^a_{1122}}^2 \\
&\quad + C^a_{1122}C^b_{1122} - C^b_{1212}C^a_{1122}), \\
A^{1S\_1}_{1111} &= 2C^a_{1212}C^a_{1111}C^b_{2222} - 2C^a_{1212}C^b_{1111}C^b_{2222} + 2C^a_{1212}{C^b_{1122}}^2 \\
&\quad + 3C^b_{1212}C^a_{1122}C^b_{1122} + 3C^b_{1212}C^a_{1111}C^a_{2222} - 5C^a_{1212}C^b_{1212}C^a_{1122} \\
&\quad - 3C^b_{1212}C^b_{1111}C^a_{2222} - 3C^b_{1212}{C^a_{1122}}^2 - 2C^a_{1212}C^a_{1122}C^b_{1122} \\
&\quad + 5C^b_{1212}C^a_{1212}C^b_{1122}, \\
A^{1S\_2}_{1111} &= C^b_{1212}\Big(3{C^b_{1122}}^2 - 4C^a_{1212}C^a_{1122} - 3C^b_{1111}C^b_{2222} - 2C^a_{1122}C^b_{1122} \\
&\quad + 4C^a_{1212}C^b_{1122} + C^a_{1111}C^a_{2222} - {C^a_{1122}}^2 + 3C^a_{1111}C^b_{2222} - C^b_{1111}C^a_{2222}\Big), \\
A^{1S\_3}_{1111} &= C^b_{1212}(C^a_{1212}C^b_{1122} - C^a_{1212}C^a_{1122} - C^a_{1122}C^b_{1122} - C^b_{1111}C^b_{2222} \\
&\quad + {C^b_{1122}}^2 + C^a_{1111}C^b_{2222}), \\
A^{2S\_0}_{1111} &= -C^a_{1122}C^b_{1122} + {C^a_{1122}}^2 - C^a_{1212}C^b_{1122} + C^a_{1212}C^a_{1122} - C^a_{1111}C^a_{2222} \\
&\quad + C^b_{1111}C^a_{2222}, \\
A^{2S\_1}_{1111} &= C^b_{1111}C^b_{2222} - C^a_{1212}C^b_{1122} + C^a_{1122}C^b_{1122} - C^a_{1111}C^b_{2222} \\
&\quad + C^a_{1212}C^a_{1122} - {C^b_{1122}}^2, \\
B^{1S\_1}_{1111} &= -C^a_{1212}\Big(4C^b_{1212}C^b_{1122} - C^b_{1111}C^b_{2222} + {C^b_{1122}}^2 - 3{C^a_{1122}}^2 + C^a_{1111}C^b_{2222} \\
&\quad - 3C^b_{1111}C^a_{2222} + 3C^a_{1111}C^a_{2222} + 2C^a_{1122}C^b_{1122} - 4C^b_{1212}C^a_{1122}\Big), \\
B^{1S\_2}_{1111} &= -2C^b_{1212}C^b_{1122}C^a_{1122} - 3C^a_{1212}{C^b_{1122}}^2 + 3C^a_{1212}C^a_{1122}C^b_{1122} \\
&\quad + 2C^b_{1212}C^b_{1111}C^a_{2222} - 5C^a_{1212}C^b_{1212}C^b_{1122} + 3C^a_{1212}C^b_{1111}C^b_{2222} \\
&\quad - 2C^b_{1212}C^a_{1111}C^a_{2222} + 5C^a_{1212}C^b_{1212}C^a_{1122} - 3C^a_{1212}C^a_{1111}C^b_{2222} \\
&\quad + 3C^b_{1212}{C^a_{1122}}^2, \\
B^{2S\_0}_{1111} &= C^a_{1111}C^a_{2222} - C^b_{1111}C^a_{2222} + C^b_{1212}C^b_{1122} - {C^a_{1122}}^2 + C^a_{1122}C^b_{1122} \\
&\quad - C^b_{1212}C^a_{1122}, \\
B^{2S\_1}_{1111} &= C^b_{1212}C^b_{1122} - C^b_{1212}C^a_{1122} - C^b_{1111}C^b_{2222} + C^a_{1111}C^b_{2222} \\
&\quad - C^a_{1122}C^b_{1122} + {C^b_{1122}}^2.
\end{aligned}
$$

Appendix C

The constants $I^{S\_a}_{2121}$, $I^{S\_b}_{2121}$, $I^{S\_a}_{211211}$, $I^{S\_b}_{211211}$, $I^{D1\_a}_{211211}$, $I^{D1\_b}_{211211}$, $I^{D2\_a}_{211211}$, $I^{D2\_b}_{211211}$ involved in the terms I^{S}_{2121}, I^{D}_{2121}, I^{S}_{211211}, I^{D1}_{211211}, I^{D2}_{211211} of the overall inertial constants I_{2121}, I_{211211}, given in (34), are:

$$
\begin{aligned}
I^{S\_a}_{2121} &= \eta C^a_{1212}C^a_{2222}C^b_{2222} + 2C^b_{1122}C^a_{2222}C^a_{1212} + 2\eta C^a_{1122}C^b_{2222}C^a_{1212} \\
&\quad + \eta^2 C^a_{1212}C^a_{2222}C^b_{2222} + 2\eta^2 C^a_{1122}C^b_{2222}C^b_{1212} + 2\eta C^b_{1122}C^a_{2222}C^b_{1212} \\
&\quad - \eta^2 C^a_{2222}C^b_{2222}C^b_{1212} - \eta C^a_{2222}C^b_{2222}C^b_{1212}, \\
I^{S\_b}_{2121} &= -2\eta C^a_{1122}C^b_{2222}C^a_{1212} - 2C^b_{1122}C^a_{2222}C^a_{1212} + \eta C^a_{1212}C^a_{2222}C^b_{2222} \\
&\quad + C^a_{1212}C^a_{2222}C^b_{2222} - C^a_{2222}C^b_{2222}C^b_{1212} - 2\eta C^b_{1122}C^a_{2222}C^b_{1212} \\
&\quad - \eta C^a_{2222}C^b_{2222}C^b_{1212} - 2\eta^2 C^a_{1122}C^b_{2222}C^b_{1212}, \\
I^{S\_a}_{211211} &= \eta\Big(10\eta C^a_{1122}C^b_{2222}C^b_{1122}C^a_{2222} + 6\eta^2 {C^a_{1122}}^2{C^b_{2222}}^2 \\
&\quad + 2\eta^3 {C^a_{1122}}^2{C^b_{2222}}^2 + \eta^4 {C^a_{1122}}^2{C^b_{2222}}^2 + 5{C^b_{1122}}^2{C^a_{2222}}^2\Big), \\
I^{S\_b}_{211211} &= 10\eta^3 C^a_{1122}C^b_{2222}C^b_{1122}C^a_{2222} + 2\eta {C^b_{1122}}^2{C^a_{2222}}^2 + {C^b_{1122}}^2{C^b_{2222}}^2 \\
&\quad + 6\eta^2 {C^b_{1122}}^2{C^a_{2222}}^2 + 5\eta^4 {C^a_{1122}}^2{C^b_{2222}}^2, \\
I^{D1\_a}_{211211} &= \eta\Big(5C^b_{1122}{C^a_{2222}}^2 + 2\eta^3 C^a_{1122}{C^b_{2222}}^2 + \eta^4 C^a_{1122}{C^b_{2222}}^2 \\
&\quad + 5\eta C^a_{2222}C^b_{2222}C^a_{1122} + 5\eta C^a_{2222}C^b_{1122}C^b_{2222} + 6\eta^2 C^a_{1122}{C^b_{2222}}^2\Big), \\
I^{D1\_b}_{211211} &= 5\eta^3 C^a_{1122}C^a_{2222}C^b_{2222} + 2\eta C^b_{1122}{C^a_{2222}}^2 + C^b_{1122}{C^a_{2222}}^2 \\
&\quad + 5\eta^4 C^a_{1122}{C^b_{2222}}^2 + 5\eta^3 C^b_{1122}C^a_{2222}C^b_{2222} + 6\eta^2 C^b_{1122}{C^a_{2222}}^2, \\
I^{D2\_a}_{211211} &= \eta(5{C^a_{2222}}^2 + \eta^4{C^b_{2222}}^2 + 2\eta^3{C^b_{2222}}^2 + 10\eta C^a_{2222}C^b_{2222} \\
&\quad + 6\eta^2{C^b_{2222}}^2), \\
I^{D2\_b}_{211211} &= 10\eta^3 C^a_{2222}C^b_{2222} + {C^a_{2222}}^2 + 6\eta^2{C^a_{2222}}^2 + 5\eta^4{C^b_{2222}}^2 + 2\eta{C^a_{2222}}^2.
\end{aligned}
$$

Appendix D

The constants $I^{S\_0}_{1111}$, $I^{S\_1}_{1111}$, $I^{S\_2}_{1111}$, $I^{S\_3}_{1111}$, $I^{S\_0}_{111111}$, $I^{S\_1}_{111111}$, $I^{S\_2}_{111111}$, $I^{S\_3}_{111111}$, $I^{S\_4}_{111111}$, $I^{S\_5}_{111111}$, $I^{D1\_0}_{111111}$, $I^{D1\_1}_{111111}$, $I^{D1\_2}_{111111}$, $I^{D1\_3}_{111111}$, $I^{D1\_4}_{111111}$, $I^{D1\_5}_{111111}$ involved in the terms I^{S}_{1111}, I^{D}_{1111}, I^{S}_{111111}, I^{D1}_{111111}, I^{D2}_{111111} of the overall inertial constants I_{1111}, I_{111111}, given in (37), take the form

$$I^{S\_0}_{1111} = -2\rho_b(B^{2S}_{1111} + 12B^{1S}_{1111}), \quad I^{S\_1}_{1111} = -48\rho_b B^{1S}_{1111} - 24\rho_a A^{1S}_{1111},$$

$$I^{S\_2}_{1111} = \rho_b {A^{1S}_{211}}^2 - 24\rho_b B^{1S}_{1111} - 48\rho_a A^{1S}_{1111},$$

$$I^{S\_3}_{1111} = \rho_a({A^{1S}_{211}}^2 - 2A^{2S}_{1111} - 24A^{1S}_{1111})$$

$$I^{S\_0}_{111111} = \rho_b(3{B^{2S}_{1111}}^2 + 40B^{2S}_{1111}B^{1S}_{1111} + 240{B^{1S}_{1111}}^2),$$

$$I^{S\_1}_{111111} = 960\rho_b {B^{2S}_{1111}}^2 + 240\rho_a {A^{1S}_{1111}}^2 + 80\rho_b B^{1S}_{1111}B^{2S}_{1111},$$

$$I^{S\_2}_{111111} = 960\rho_a {A^{1S}_{1111}}^2 + 1440\rho_b {B^{1S}_{1111}}^2 + 80\rho_b B^{1S}_{1111}B^{2S}_{1111},$$

$$I^{S\_3}_{111111} = 960\rho_b {B^{1S}_{1111}}^2 + 1440\rho_a {A^{1S}_{1111}}^2 + 40\rho_a A^{1S}_{1111}A^{2S}_{1111},$$

$$I^{S\_4}_{111111} = 240\rho_b {B^{1S}_{1111}}^2 + 960\rho_a {A^{1S}_{1111}}^2 + 80\rho_a A^{1S}_{1111}A^{2S}_{1111},$$

$$I^{S\_5}_{111111} = \rho_a(3{A^{2S}_{1111}}^2 + 40A^{2S}_{1111}A^{1S}_{1111} + 240{A^{1S}_{1111}}^2),$$

$$\begin{aligned} I^{D1\_0}_{111111} = {} & \rho_b(20B^{2S}_{1111}B^{1D}_{1111} + 20B^{2D}_{1111}B^{1S}_{1111} + 240B^{1S}_{1111}B^{1D}_{1111} \\ & +3B^{2S}_{1111}B^{2D}_{1111}), \end{aligned}$$

$$\begin{aligned} I^{D1\_1}_{111111} = {} & 40(\rho_b B^{2S}_{1111}B^{1D}_{1111} + 24\rho_b B^{1D}_{1111}B^{1S}_{1111} + 6\rho_a A^{1S}_{1111}A^{1D}_{1111} \\ & +\rho_b B^{2S}_{1111}B^{1D}_{1111}), \end{aligned}$$

$$\begin{aligned} I^{D1\_2}_{111111} = {} & 20(\rho_b B^{2S}_{1111}B^{1D}_{1111} + \rho_b B^{2D}_{1111}B^{1S}_{1111} + 72\rho_b B^{1S}_{1111}B^{1D}_{1111} \\ & +48\rho_a A^{1S}_{1111}A^{1D}_{1111}), \end{aligned}$$

$$\begin{aligned} I^{D1\_3}_{111111} = {} & 20(48\rho_b B^{1S}_{1111}B^{1D}_{1111} + 72\rho_a A^{1S}_{1111}A^{1D}_{1111} + \rho_a A^{2S}_{1111}A^{1D}_{1111} \\ & +\rho_a A^{1S}_{1111}A^{2D}_{1111}), \end{aligned}$$

$$\begin{aligned} I^{D1\_4}_{111111} = {} & 40(6\rho_b B^{1S}_{1111}B^{1D}_{1111} + 24\rho_a A^{1S}_{1111}A^{1D}_{1111} + \rho_a A^{2S}_{1111}A^{2D}_{1111} \\ & +\rho_a A^{2S}_{1111}A^{1D}_{1111}), \end{aligned}$$

$$\begin{aligned} I^{D1\_5}_{111111} = {} & \rho_a(3A^{2S}_{1111}A^{2D}_{1111} + 20A^{2S}_{1111}A^{1D}_{1111} + 20A^{1S}_{1111}A^{2D}_{1111} \\ & +240A^{1S}_{1111}A^{1D}_{1111}), \end{aligned}$$

and $A^{1S}_{211} = \frac{C^b_{1122} - C^a_{1122}}{C^a_{2222} + \eta C^b_{2222}}$, $A^{1D}_{1111} = A^{1D}_{1122}$, $A^{2D}_{1111} = A^{2D}_{1122}$, $B^{1D}_{1111} = B^{1D}_{1122}$, $B^{2D}_{1111} = -\eta \frac{C^a_{1212}}{C^b_{1212}} A^{2D}_{1122}$ and the coefficients A^{1S}_{1111}, A^{2S}_{1111}, B^{1S}_{1111}, B^{2S}_{1111}, A^{1D}_{1122}, A^{2D}_{1122}, B^{1D}_{1122} are previously given in Appendix B.

Appendix E

The dispersive functions for waves travelling along the layering direction of the layered material represented in the diagrams of Fig. 3 are here represented for the non-dimensional wave-number $k\varepsilon \in [0, 3\pi]$ in the diagrams of Fig. E.1. This analysis, that is extended to non-dimensional wave-numbers $k\varepsilon \geqslant \pi$ (i.e. for wavelength λ comparable with the characteristic size ε), is aimed to verify the accuracy of the results obtained from the dynamic homogenizing model here proposed compared to the exact curves obtained by the Floquet–Bloch theory. It is worth to note that in this range, the smaller and the larger scales are becoming comparable. This outcome is in agreement with the conclusions obtained in Smyshlyaev and Cherednichenko (2000), where the variational-asymptotic method is shown to provide good results even when the parameter ε is small but not too small.

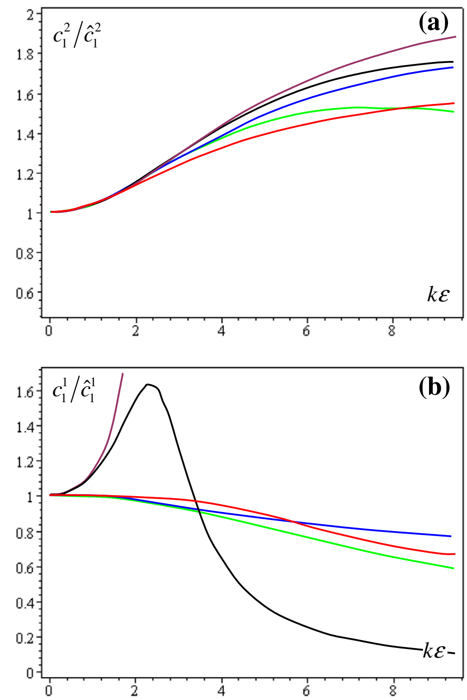

Fig. E.1. Shear (a) and compressional (b) waves along the layering direction e_1: non-dimensional phase velocities versus non-dimensional wave-number. Red line: heterogeneous material (Sun et al., 1968); Green line: proposed model; Blue line: simplified proposed model; Violet line: Bacigalupo and Gambarotta (2012a); Black line: Bacigalupo and Gambarotta (2012b). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

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